Lecture in Les Houches School on KPZ

Introduction to the KPZ equation and its experimental aspects

ver.3

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Chapter 1

Introduction

- why should we care this? -

Physics of Critical Phenomena & Scaling Laws

Equilibrium (major player: Ising)

1869	Discovery of liquid-vapor critical point (Ising class)
1890's-	$\beta \approx 0.3\text{-}0.4$ (cf. 3D Ising $\beta \approx 0.326$)
1944	Onsager's solution to 2D Ising
1950's-	Experiments on binary fluids & Ising-type magnets
1971	Wilson's renormalization group, ϕ^4 model (continuum equation) "Ising universality class"
1984	2D conformal field theory classifying universality classes
2011-	Conformal approach to 3D Ising
⋮	⋮

Non-eq (major player: KPZ?)

1980's	Scaling laws for discrete models of interface growth
1986	KPZ eq. (continuum eq.)
1997	Experiments on KPZ exponents
2000	Exact solutions to 1D discrete models on distributions & correlations
2010	Experiment on exact results
2010	Exact solutions to 1D KPZ eq.
2019	KPZ corr. func. in Heisenberg
2021-22	Exp'ts on KPZ-Heisenberg link
2022	Exp't on KPZ in polaritons
⋮	⋮

Classic Target of KPZ: Growing Interfaces

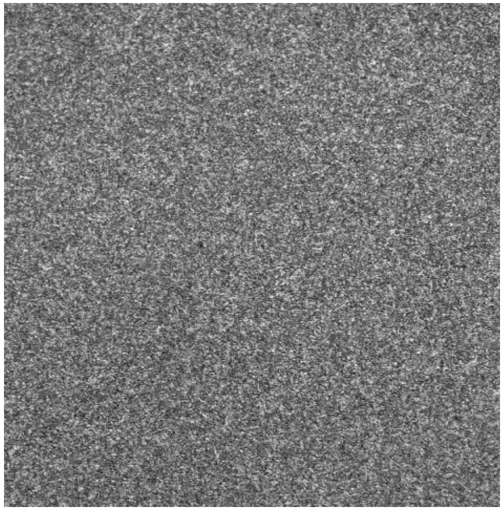
fire spreading

Calahorra Mountain Club's Facebook [1]

paper combustion

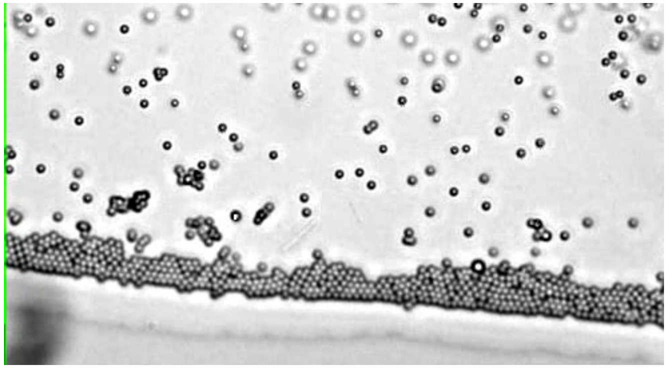
Timonen group,
PRL 1997, PRE 2001 [2,3]

liquid crystal turbulence



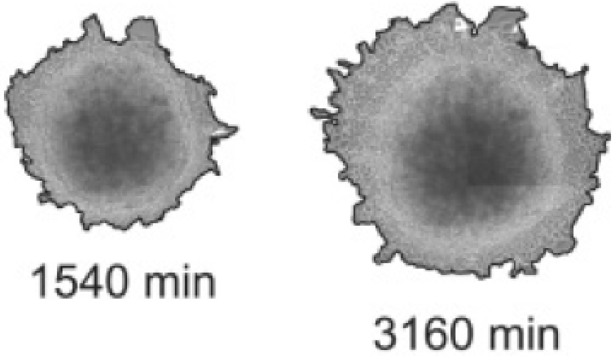
Takeuchi *et al.*, 2010-12 [4-6]

particle deposition



Yunker *et al.* Nature 2011 [7] (see also [8])

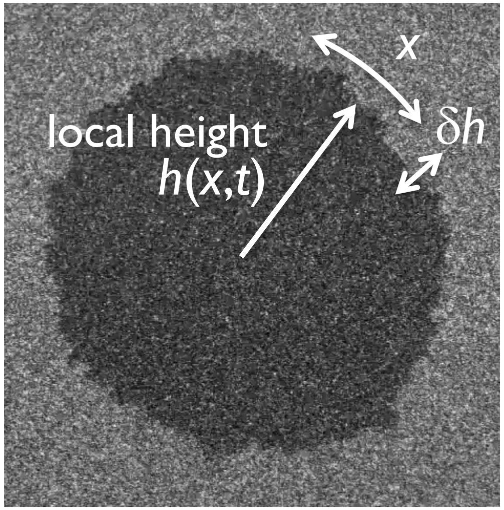
cancer cell proliferation



Huergo *et al.* Phys. Rev. E 2012 [9]

Classic Target of KPZ: Growing Interfaces

liquid crystal turbulence

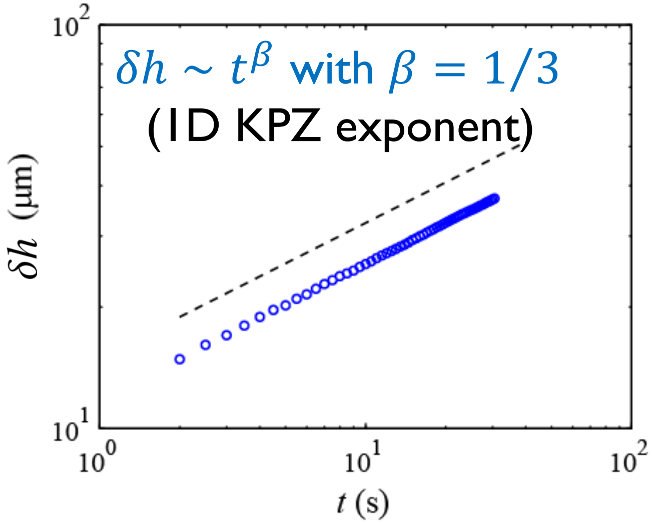


Takeuchi *et al.*, 2010-12 [4-6]

$\delta h \equiv h - \langle h \rangle$

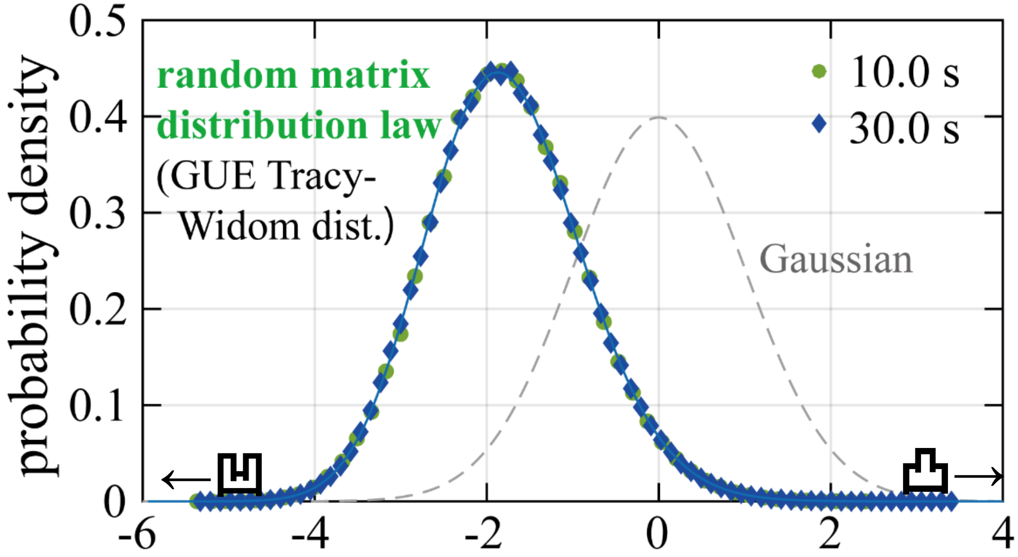
scaling law

fluctuation amplitude δh vs time t



distribution law

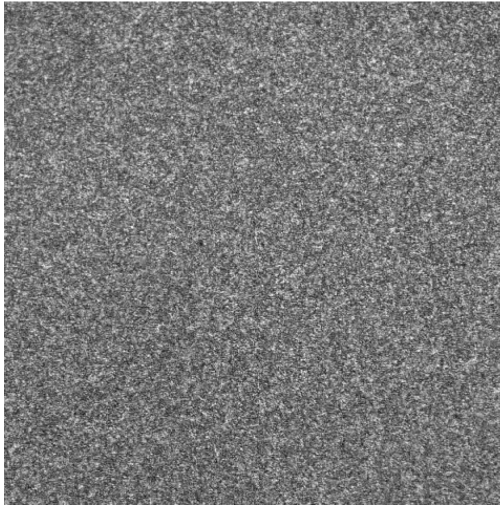
(outcome of exact solution)



Expanding Scope of KPZ

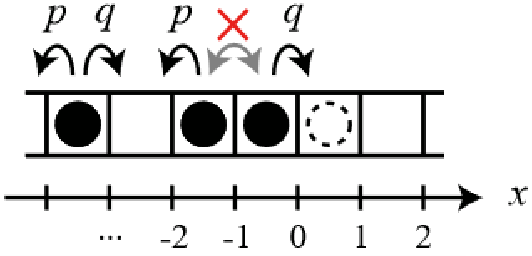
Physical phenomena / models

growing interfaces

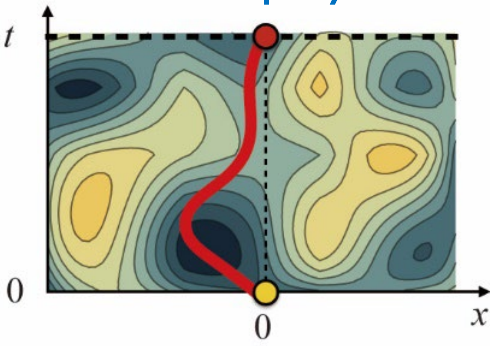


Takeuchi *et al.*, 2010-12 [4-6]

stochastic particle transport



directed polymer



stirred fluid

interacting Bose gas

nonlinear fluctuating hydrodynamics

Anderson insulator

quantum spin chains

polariton condensate

Theoretical concepts

universal scaling laws out of equilibrium (& in), integrable systems, random matrix theory, probability theory, combinatorics, ...

Lecture Plan

Aims

- To understand (1) what is KPZ, (2) connections to different systems, (3) main outcomes of the modern developments.
- Foster intuitive understanding of the outcomes,
  rather than technical & mathematical details.

Table of contents

Recommended Reviews on KPZ

(Takeuchi's lecture notes)
[10] K.A. Takeuchi, Physica A 504, 77 (2018)

(before 2000 = before the exact solutions)
[11] A.-L. Barabási & H. E. Stanley, *Fractal Concepts in Surface Growth* (Cambridge Univ. Press, 1995).
[12] T. Halpin-Healy & Y.-C. Zhang, Phys. Rep. 1995.
[13] J. Krug, Adv. Phys. 1997.

(after 2000)
[14] I. Corwin, Random Matrices Theory Appl. 2012.
[15] J. Quastel & H. Spohn, J. Stat. Phys. 2015.
[16] T. Kriecherbauer and J. Krug, J. Phys. A 2010.
[17] T. Sasamoto, Prog. Theor. Exp. Phys. 2016.
[18] H. Spohn, Lect. Notes Phys. 2016 (nonlinear fluctuating hydrodynamics)
[19] I.Corwin & H. Shen, Bull. Am. Math. Soc. 2020 (well-definedness of KPZ eq.)
[20] V. B. Bulchandani et al., J. Stat. Mech. 2021 (quantum spin chains)
[21] S. Prolhac, arXiv:2401.15016 (KPZ in finite systems)

Chapter 2

Scaling exponents and universality classes

Main references [10-13]

Classic Target of KPZ: Growing Interfaces

Coarse-grained time evolution for those growing interfaces?

fire spreading

Calahorra Mountain Club's Facebook [1]

paper combustion

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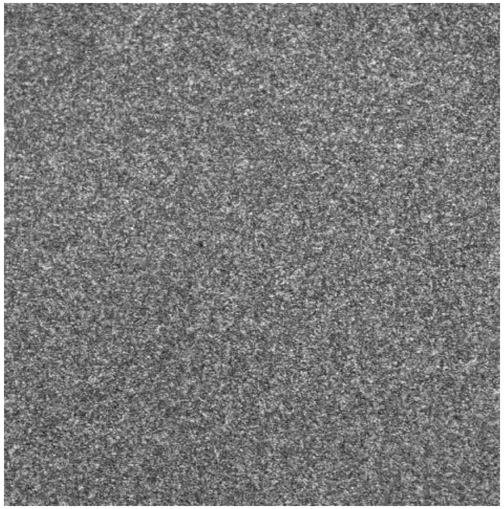
PRL 1997, PRE 2001 [2,3]

liquid crystal turbulence



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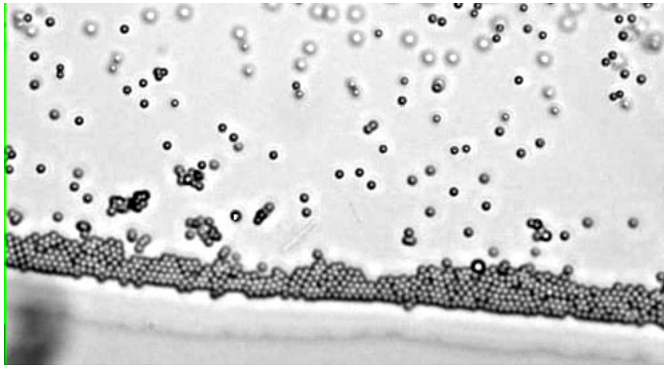

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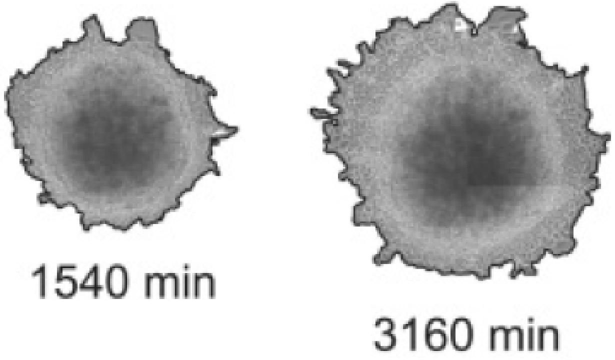
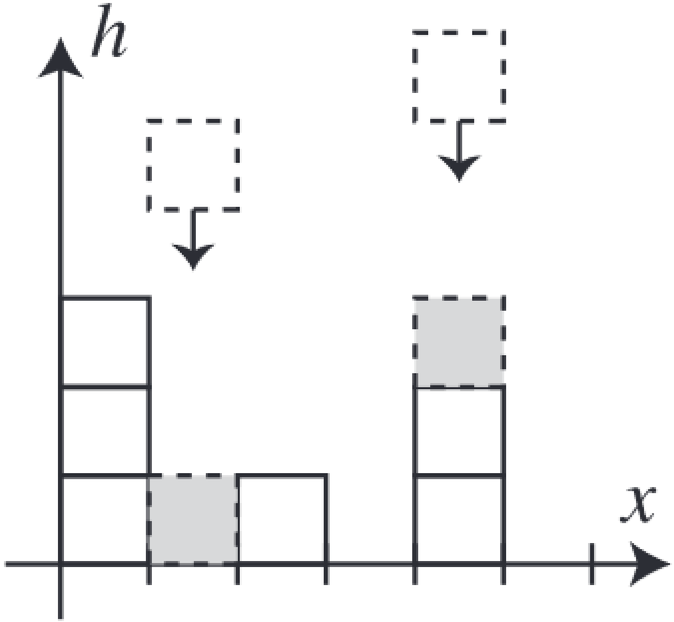


Huergo *et al.* Phys. Rev. E 2012 [9]

A Warm-Up Example

Random deposition of blocks

- Drop a block randomly at a constant rate at random positions.
- Blocks just accumulate. No interaction with neighbor sites.



➔ $\langle h(x,t)\rangle \sim t$

$$\delta h(x,t) \equiv h(x,t) - \langle h(x,t)\rangle$$

$\sim t^{1/2}$ ($\because$ law of large numbers), Gaussian

Coarse-graining

➔ $$\boxed{\frac{\partial}{\partial t}h(x,t) = v_0 + \eta(x,t)}$$

$\eta(x,t)$: white Gaussian noise

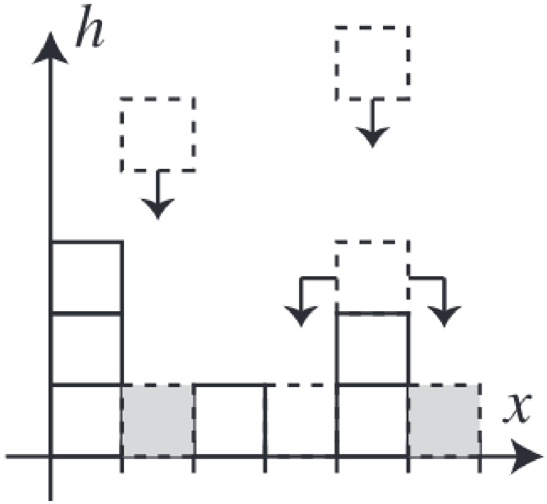
$\langle \eta(x,t)\rangle = 0$

$\langle \eta(x,t)\eta(x',t')\rangle = D\delta(x-x')\delta(t-t')$

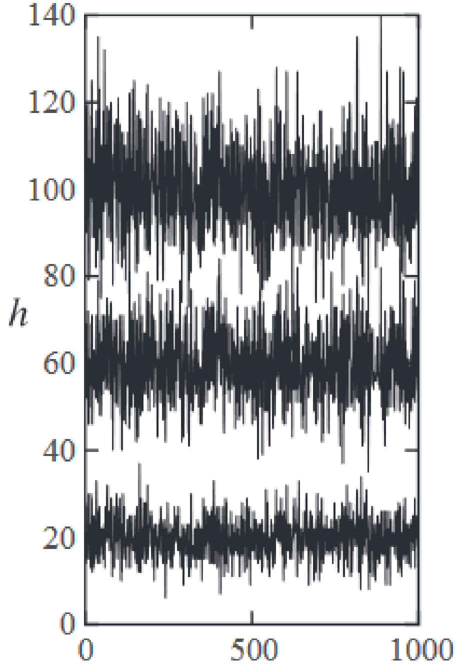
If Blocks Interact…

In the case of random deposition with surface relaxation

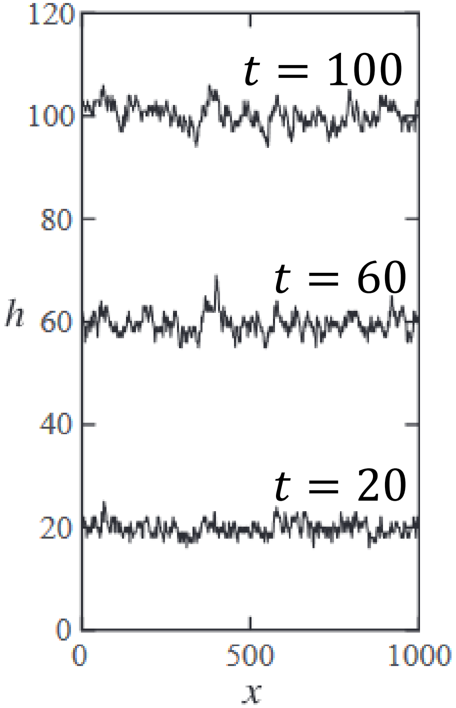
- Dropped block can slide to its lower neighbor.



w/o surface relaxation



w/ surface relaxation

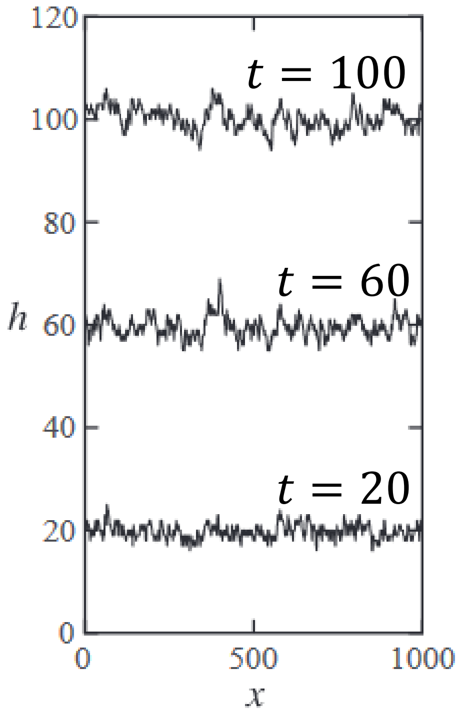


Coarse-graining

➔ $$\boxed{\frac{\partial}{\partial t}h(x,t) = v_0 + \nu\nabla^2 h + \eta(x,t)}$$

"Edwards-Wilkinson equation"

If Blocks Interact…



$$\boxed{\frac{\partial}{\partial t} h(x,t) = v_0 + \nu\nabla^2 h + \eta(x,t)}$$

"Edwards-Wilkinson equation"

- v_0 can be omitted by $h \equiv h' + v_0 t$
- Scaling law?

Suppose solutions are statistically invariant under the following scale transformations:

$$\boxed{x \equiv bx',\ t \equiv b^z t',\ \delta h \equiv b^\alpha \delta h'} \Rightarrow \boxed{\delta h \sim t^\beta \text{ with } \beta = \frac{\alpha}{z}}$$

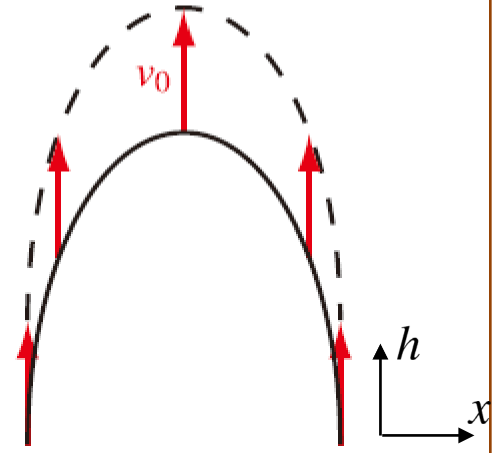
(scale invariance, or more specifically, self-affinity)

➔ $b^{\alpha - z} = b^{\alpha - 2} = b^{-(d+z)/2}$ (d: space dimensionality)

$\therefore \boxed{z = 2,\ \alpha = \frac{2-d}{2},\ \beta = \frac{2-d}{4}}$ Edwards-Wilkinson universality class

$d = 1$ ➔ $\alpha = 1/2, \beta = 1/4 \therefore \delta h \sim t^{1/4}$

Call for Nonlinearity



Edwards-Wilkinson equation

$$\boxed{\frac{\partial}{\partial t}h(x,t) = v_0 + \nu\nabla^2 h + \eta(x,t)}$$

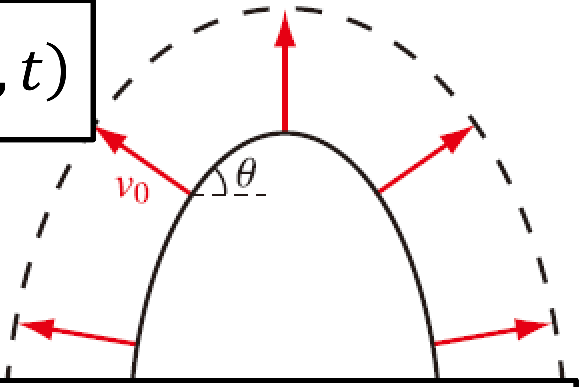
linear & easy to solve exactly!

… but unnatural.

perhaps more naturally

Kardar-Parisi-Zhang (KPZ) equation (1986) [22]

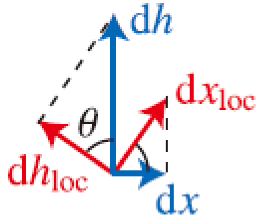
$$\boxed{\frac{\partial}{\partial t}h(x,t) = v_0 + \nu\nabla^2 h + \frac{\lambda}{2}(\nabla h)^2 + \eta(x,t)}$$



- Generic equation incorporating lowest-order nonlinearity.
- For $d = 1$, (to show later)

$$\boxed{\alpha = \frac{1}{2},\ \beta = \frac{1}{3},\ z = \frac{3}{2}}$$

- KPZ universality class.



$$\tan\theta = |\nabla h|,$$

$$\mathrm{d}h = \frac{\mathrm{d}h_{\mathrm{loc}}}{\cos\theta} = \mathrm{d}h_{\mathrm{loc}}\sqrt{1 + (\nabla h)^2}$$

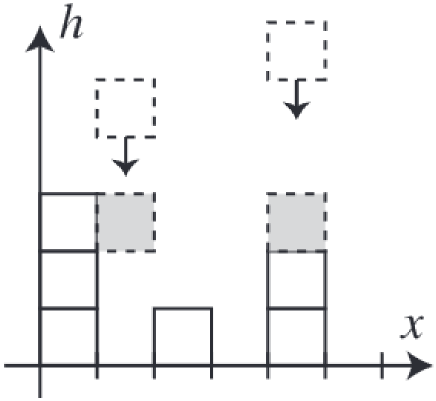
Some Remarks on KPZ

KPZ equation $$\boxed{\frac{\partial}{\partial t}h(x,t) = v_0 + \nu\nabla^2 h + \frac{\lambda}{2}(\nabla h)^2 + \eta(x,t)}$$

- v_0 can be omitted by $h \equiv h' + v_0 t$
- Symmetry: KPZ is invariant under
  - Time translation $t \equiv t' + t_0$
  - Space translation $x \equiv x' + x_0$
  - Space inversion, e.g., $x \equiv -x'$ & space rotation
  - Height translation $h \equiv h' + h_0$ ➔ term like $-ah$ is forbidden.

Systems are automatically at criticality.

- KPZ class generically arises under this symmetry.
  - No isotropic growth needed.



Example:
ballistic deposition

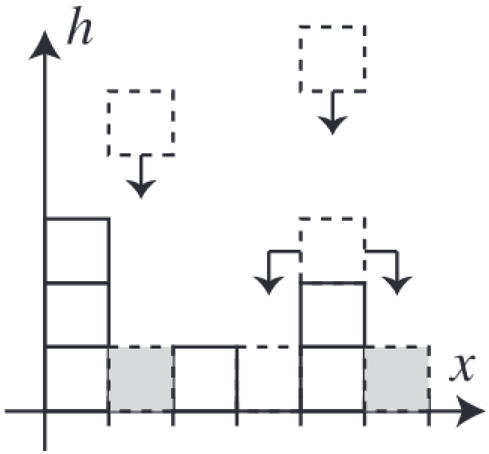
Other Universality Classes

Quenched KPZ equation

$$\boxed{\frac{\partial}{\partial t}h(x,t) = F + \nu\nabla^2 h + \frac{\lambda}{2}(\nabla h)^2 + \eta(x, h(x,t))}$$

- F can NOT be omitted
  ($\because$ by $h \equiv h' + Ft,\ \eta$ is not quenched any more)
- Pinning-depinning transition
  - $F > F_c$: interface grows, KPZ scaling ($\eta(x,h)$ is equivalent to $\eta(x,t)$)
  - $F < F_c$: interface pinned
  - $F \approx F_c$: critical scaling, $\alpha = \beta \approx 0.633.$
    "quenched KPZ class" related to the directed percolation class [11,23,24]

Other Universality Classes



Conserved growth [11]

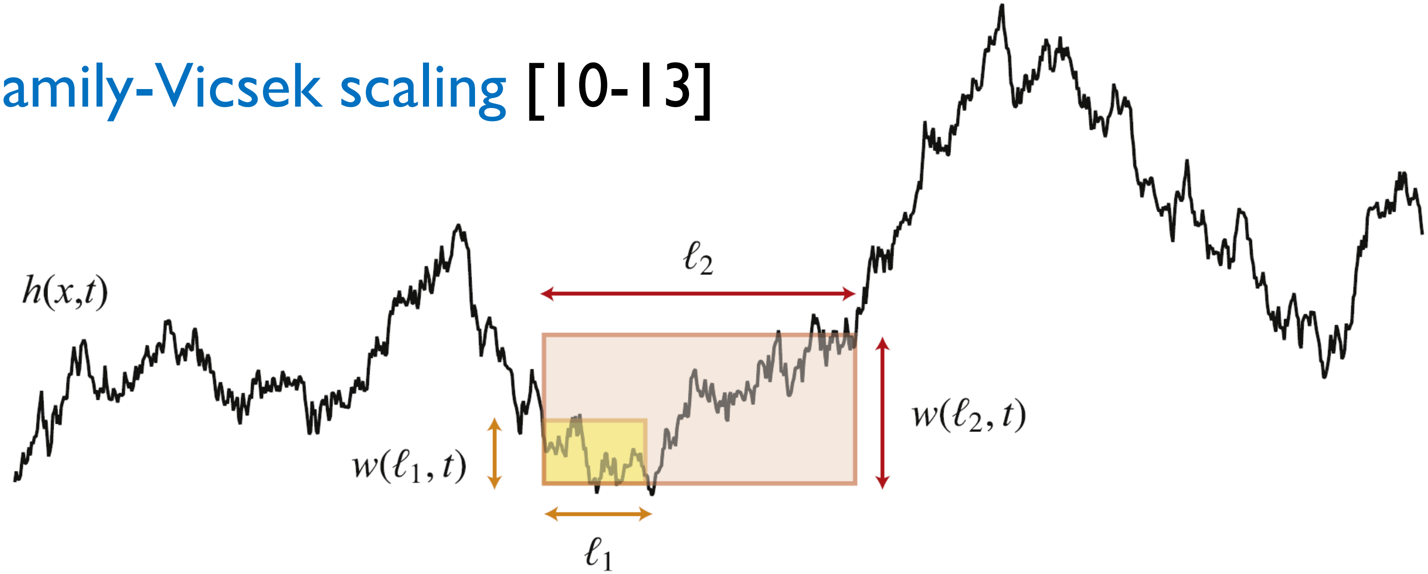
- Suppose surface diffusion of particles takes place much faster than deposition.
- Then we have: $\boxed{\frac{\partial h}{\partial t} = -\nabla \cdot J + v_0 + \eta(x,t)}$ J: flux
  v_0 can be omitted
- Simplest case: $J \propto -\nabla(\text{chemical potential } \mu), \mu \propto -\nabla^2 h$
  $\rightarrow \boxed{\frac{\partial h}{\partial t} = -\kappa\nabla^4 h + \eta(x,t)}$: Mullins-Herring (MH) equation
  - $\alpha = \frac{4-d}{2}, \beta = \frac{4-d}{8}, z = 4$ (MH class)
- Nonlinear case: molecular beam epitaxy (MBE) class

$$\boxed{\frac{\partial h}{\partial t} = -\kappa\nabla^4 h + \lambda_1 \nabla^2 (\nabla h)^2 + \lambda_2 \nabla \cdot (\nabla h)^3 + \eta(x,t)}$$

  - Renormalization group result: $\alpha = \frac{4-d}{3}, \beta = \frac{4-d}{8+d}, z = \frac{8+d}{3}$

How to Measure the Exponents?

Family-Vicsek scaling [10-13]



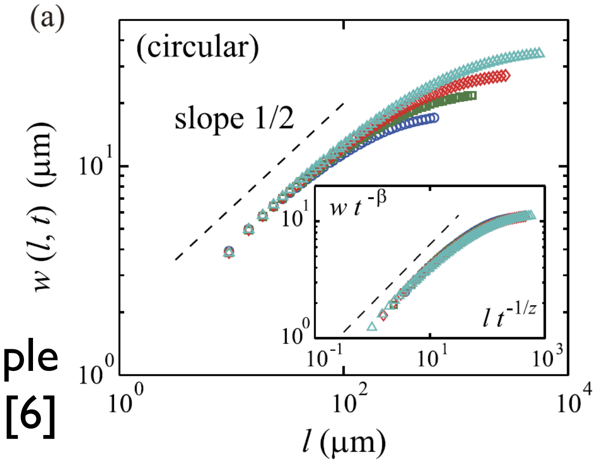
Measure a scale of δh, as a function of lateral scale ℓ & time t.

e.g., interface width $w(\ell,t)$ = standard deviation in length ℓ

Then, Family-Vicsek scaling

$$w(\ell,t) \sim \begin{cases} \ell^{\alpha} & (\ell \ll \xi(t)) \\ t^{\beta} & (\ell \gg \xi(t)) \end{cases}$$

with $\xi(t) \sim t^{1/z}$



Example from [6]

Chapter 3

Basic Properties of the KPZ Equation

Main references [10-13]

3.1 Relation to the Noisy Burgers Equation

KPZ equation $\boxed{\frac{\partial}{\partial t}h(\vec{x},t) = \cancel{v_0} + \nu\nabla^2 h + \frac{\lambda}{2}\left(\vec{\nabla}h\right)^2 + \eta(\vec{x},t)}$

Take the gradient & define $\vec{v}(\vec{x},t) \equiv -\lambda\vec{\nabla}h$

$\rightarrow \boxed{\frac{\partial\vec{v}}{\partial t} + \left(\vec{v}\cdot\vec{\nabla}\right)\vec{v} = \nu\nabla^2\vec{v} - \lambda\vec{\nabla}\eta}$: noisy Burgers equation [25]

(toy model for fluid & shock waves)

Consequences

- Invariant under Galilean trans. $\boxed{\vec{v}'(\vec{x}-\vec{v}_0 t, t) \equiv \vec{v}(\vec{x},t) - \vec{v}_0}$
- Galilean symmetry is kept under scale transformation.

  → Advection term $\left(\vec{v}\cdot\vec{\nabla}\right)\vec{v}$ is non-renormalized.

- If $\vec{v}(\vec{x},t) \equiv -\vec{\nabla}h \rightarrow \frac{\partial\vec{v}}{\partial t} + \lambda\left(\vec{v}\cdot\vec{\nabla}\right)\vec{v} = \nu\nabla^2\vec{v} - \vec{\nabla}\eta$

  → λ is invariant under scale transformation.

- Scale transformation $x \equiv bx',\ t \equiv b^z t',\ \delta h \equiv b^\alpha \delta h'$

  → $b^{\alpha-z} = b^{2(\alpha-1)} \rightarrow \boxed{\alpha + z = 2}$ valid for any d!

3.2 Stationary State of 1D KPZ Equation

Langevin equation & Fokker-Planck equation [26]

$$\boxed{\frac{dX_i}{dt} = F_i\left[\{X_j\}\right] + \eta_i(t)}$$ with $\langle \eta_i(t) \rangle = 0,\ \left\langle \eta_i(t)\eta_j(t') \right\rangle = D\delta_{ij}\delta(t-t')$

$$\boxed{\frac{\partial}{\partial t}P[\{X_j\},t] = -\sum_i \frac{\partial}{\partial X_i}F_i[\{X_j\}]P[\{X_j\},t] + \frac{D}{2}\sum_i \frac{\partial^2}{\partial X_i^2}P[\{X_j\},t]}$$

Langevin PDE & functional Fokker-Planck equation [26]

$$\boxed{\frac{\partial}{\partial t}h(\vec{x},t) = F[h(\vec{x},t)] + \eta(\vec{x},t)}$$

$$\boxed{\frac{\partial}{\partial t}P[h(\vec{x}),t] = -\int d^d\vec{x}\,\frac{\delta}{\delta h}F[h(\vec{x})]P[h(x),t] + \frac{D}{2}\int d^d\vec{x}\,\frac{\delta^2}{\delta h^2}P[h(\vec{x}),t]}$$

with functional derivative $\dfrac{\delta}{\delta h} \equiv \dfrac{\partial}{\partial h} - \vec{\nabla}\cdot\dfrac{\partial}{\partial(\vec{\nabla}h)}$

3.2 Stationary State of 1D KPZ Equation

Edwards-Wilkinson equation (d dimension)

$$\boxed{\frac{\partial}{\partial t}h(\vec{x},t) = \nu\nabla^2 h + \eta(\vec{x},t)}$$

$$\boxed{\frac{\partial}{\partial t}P[h(\vec{x}),t] = -\int d^d\vec{x}\frac{\delta}{\delta h}(\nu\nabla^2 h)P[h(\vec{x}),t] + \frac{D}{2}\int d^d\vec{x}\frac{\delta^2}{\delta h^2}P[h(\vec{x}),t]}$$

Stationary solution $\left(\frac{\partial P}{\partial t} = 0\right)$

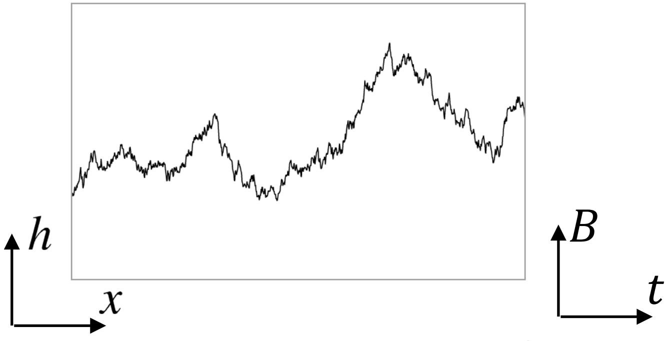
- $P_{\mathrm{stat}}^{\mathrm{EW}}[h(\vec{x})] \propto \exp\left[-\int d^d x \frac{\nu}{D}\left(\vec{\nabla}h\right)^2\right]$
- For 1D: Stationary solution = Brownian motion

$\boxed{h_{\mathrm{stat}}^{\mathrm{EW,1D}}(x) = \sqrt{A}B(x)}$ with $A \equiv \frac{D}{2\nu}$

$B(x)$ = standard Brownian motion

$$\langle[B(t+\Delta t) - B(t)]^2\rangle = \Delta t$$

$$\rightarrow \Delta h \sim \Delta x^{1/2} \quad \therefore \alpha = \frac{1}{2}$$



for interface

for BM

3.2 Stationary State of 1D KPZ Equation

KPZ equation (d dimension)

$$\boxed{\frac{\partial}{\partial t}h(\vec{x},t) = \nu\nabla^2 h + \frac{\lambda}{2}\left(\vec{\nabla}h\right)^2 + \eta(\vec{x},t)}$$

$$\boxed{\frac{\partial P}{\partial t} = -\int d^d\vec{x}\frac{\delta}{\delta h}(\nu\nabla^2 h + \frac{\lambda}{2}\left(\vec{\nabla}h\right)^2)P[h(\vec{x}),t] + \frac{D}{2}\int d^d\vec{x}\frac{\delta^2}{\delta h^2}P[h(\vec{x}),t]}$$

Stationary solution ($\frac{\partial P}{\partial t} = 0$)

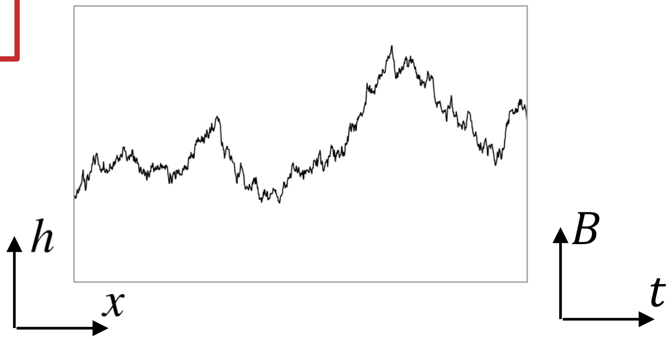
- Not available for general d.
- Exceptionally, for 1D, stationary solution = Brownian motion

$$\boxed{h_{\mathrm{stat}}^{\mathrm{KPZ,1D}}(x) = h_{\mathrm{stat}}^{\mathrm{EW,1D}}(x) = \sqrt{A}B(x)}$$

$\therefore \alpha = 1/2.$

- With $\alpha + z = 2$,

$$\boxed{\alpha = \frac{1}{2},\ \beta = \frac{1}{3},\ z = \frac{3}{2}}$$ (1D KPZ)



for interface

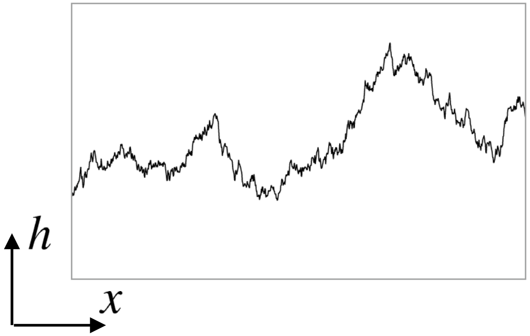
for BM

3.3 Well-definedness of KPZ Equation [19]

KPZ equation

$$\boxed{\frac{\partial}{\partial t}h(\vec{x},t) = \nu\nabla^2 h + \frac{\lambda}{2}\left(\vec{\nabla}h\right)^2 + \eta(\vec{x},t)}$$

differentiate! discontinuous everywhere!



- KPZ equation (as is) is ill-defined!
  - Problem circumvented in discretized KPZ equation, but its interpretation is not so trivial (see [27] for simulations of discretized KPZ equation).
- What do we mean by "KPZ equation"? (what is its solution?)
- The answer is established for 1D [19]
  - Cole-Hopf approach [17,19,28,29] (to describe below)
  - Hairer's rough path approach [30-33]

3.3 Well-definedness of KPZ Equation [17,19,28,29]

KPZ equation $\boxed{\frac{\partial}{\partial t} h(\vec{x},t) = \nu \nabla^2 h + \frac{\lambda}{2}\left(\vec{\nabla} h\right)^2 + \eta(\vec{x},t)}$

- Cole-Hopf transformation: $\boxed{Z(x,t) \equiv \exp\left[\frac{\lambda}{2\nu} h(x,t)\right]}$
- If we use the usual chain rule ➜ stochastic heat equation

$$\boxed{\frac{\partial}{\partial t} Z(x,t) = \nu \nabla^2 Z(x,t) + \frac{\lambda}{2\nu} Z(x,t) \text{ "}\times\text{" } \eta(x,t)}$$

- Nonlinearity disappears! ... at the cost of multiplicative noise.
- Various definitions of the multiplicative noise term exist [26]

For $\frac{dZ}{dt} = F(Z,t) + G(Z,t)\text{"}\times\text{"}\eta(t)$ or $dZ = F(Z,t)dt + G(Z,t)\text{"}\times\text{"}\underline{dB(t)}$

$$dB(t) \equiv B(t+dt) - B(t)$$

  - Itô product: $G(Z,t)dB(t) \equiv \lim_{\Delta t \to 0} G(Z(t_i),t_i)[B(t_{i+1}) - B(t_i)]$
  - Stratonovich: $G(Z,t) \circ dB(t) \equiv \lim_{\Delta t \to 0} G(\frac{Z(t_{i+i})+Z(t_i)}{2}, t_i)[B(t_{i+1}) - B(t_i)]$
- The usual chain rule is valid only for the Stratonovich product [26].

3.3 Well-definedness of KPZ Equation [17,19,28,29]

KPZ equation $\boxed{\frac{\partial}{\partial t}h(\vec{x},t) = \nu\nabla^2 h + \frac{\lambda}{2}\left(\vec{\nabla}h\right)^2 + \eta(\vec{x},t)}$

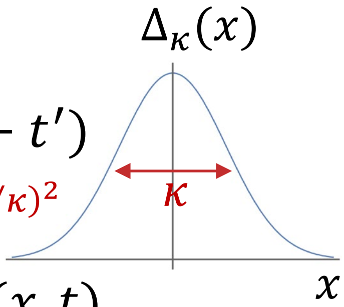
$$\Big\downarrow \; Z \equiv e^{\frac{\lambda}{2\nu}h}$$

- Stochastic heat equation (Stratonovich)

$$\boxed{\frac{\partial}{\partial t}Z(x,t) = \nu\nabla^2 Z(x,t) + \frac{\lambda}{2\nu}Z(x,t)\circ\eta(x,t)}$$

  - ➢ Equivalent to the KPZ equation (as is). Ill-defined!

- Let's consider smoothed noise $\eta_\kappa(x,t)$



  - ➢ $\langle\eta_\kappa(x,t)\rangle = 0, \langle\eta_\kappa(x,t)\eta_\kappa(x',t')\rangle = D\underline{\Delta_\kappa(x-x')}\delta(t-t')$

    $\Delta_\kappa(x) \equiv \frac{1}{\kappa\sqrt{\pi}}e^{-(x/\kappa)^2}$
  - ➢ Itô-Stratonovich conversion [26]

    $Z(x,t)\circ dB_\kappa(x,t) = Z(x,t)dB_\kappa(x,t) + \frac{1}{2}dZ(x,t)dB_\kappa(x,t)$
  - ➢ $\langle dZ(x,t)dB_\kappa(x,t)\rangle = \frac{\lambda}{2\nu}\langle Z(x,t)\rangle\underline{\langle dB_\kappa(x,t)dB_\kappa(x,t)\rangle}$

    $= D\Delta_\kappa(0)dt \propto 1/\kappa \xrightarrow{\kappa\to 0} \infty$!
  - ➢ But it's not too bad… we just have a constant drift $\propto \frac{1}{\kappa} \to \infty$

3.3 Well-definedness of KPZ Equation [17,19,28,29]

KPZ equation $\boxed{\frac{\partial}{\partial t} h(\vec{x},t) = \nu \nabla^2 h + \frac{\lambda}{2}\left(\vec{\nabla} h\right)^2 + \eta(\vec{x},t)}$

$\downarrow \; Z \equiv e^{\frac{\lambda}{2\nu}h}$

- Stochastic heat equation (Stratonovich)

$$\boxed{\frac{\partial}{\partial t} Z(x,t) = \nu \nabla^2 Z(x,t) + \frac{\lambda}{2\nu} Z(x,t) \circ \eta(x,t)}$$

  - Equivalent to the KPZ equation (as is). Ill-defined!
  - Has a constant drift $\propto \frac{1}{\kappa} \to \infty$ which doesn't exist in the Itô form.

- Stochastic heat equation (Itô)

$$\boxed{\frac{\partial}{\partial t} Z(x,t) = \nu \nabla^2 Z(x,t) + \frac{\lambda}{2\nu} Z(x,t)\eta(x,t)}$$

  - Well-defined! (even mathematically) [19,28,29]
  - $\boxed{h(x,t) \coloneqq \frac{2\nu}{\lambda} \log Z(x,t)}$: the "solution of the KPZ equation"

Chapter 4

Experiments on KPZ and related interfaces

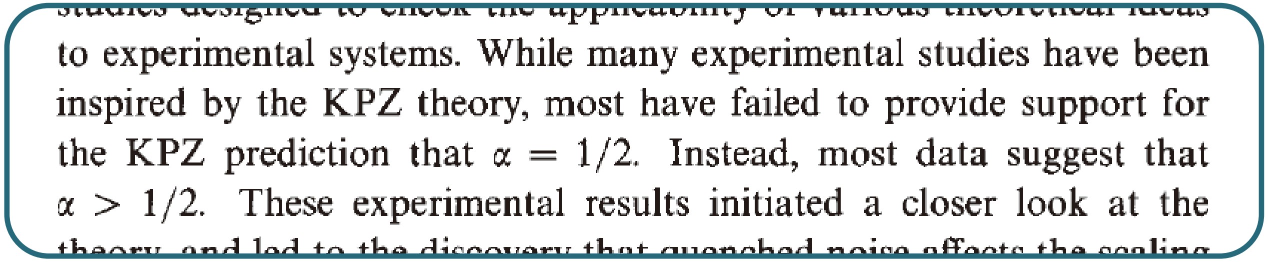
Main references [34]

(Takeuchi's survey of experiments on KPZ & directed percolation)

First of All…

KPZ is NOT so often encountered in real world interfaces, yet more & more examples have been reported recently. [34]

from Barabási & Stanley's textbook (1995) [11]



studies designed to check the applicability of various theoretical ideas to experimental systems. While many experimental studies have been inspired by the KPZ theory, most have failed to provide support for the KPZ prediction that $\alpha = 1/2$. Instead, most data suggest that $\alpha > 1/2$. These experimental results initiated a closer look at the theory, and led to the discovery that quenched noise affects the scaling

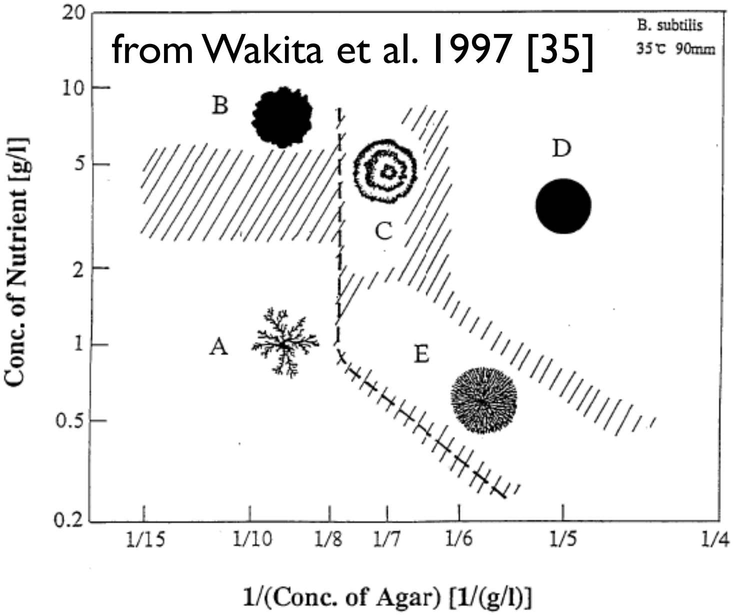
Some nontrivial requirements for KPZ

- Short-range interaction
- Large enough system size
- Short-time memory

Long-range effect may generate fractal (not self-affine) patterns (e.g, snowflake, viscous fingering)

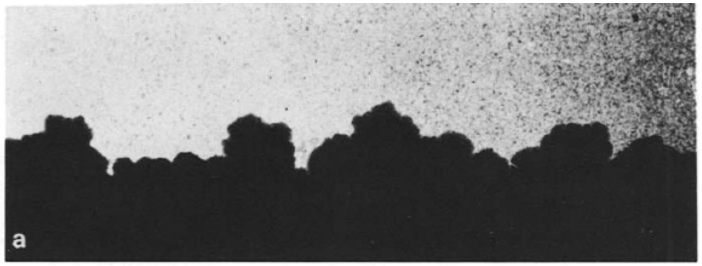
Now let's see actual experiments.

Bacteria Colony



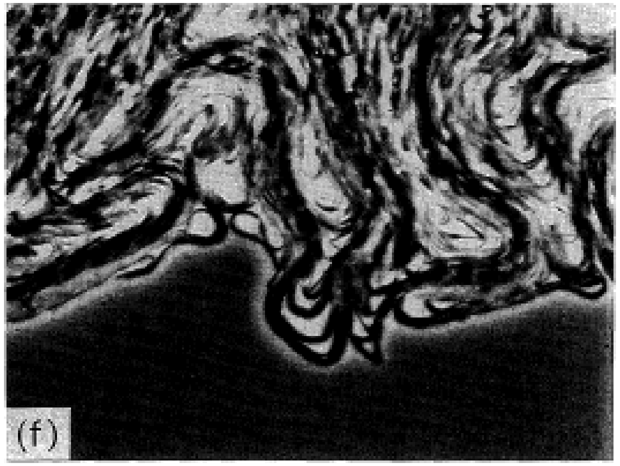
from Wakita et al. 1997 [35]

E. coli ⇨ $\alpha \approx 0.78$ (Vicsek et al. 1990 [36])

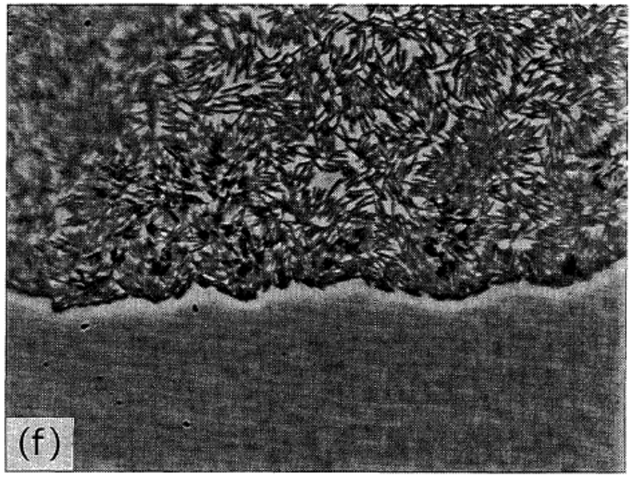


B. subtilis (Wakita et al. 1997 [35])

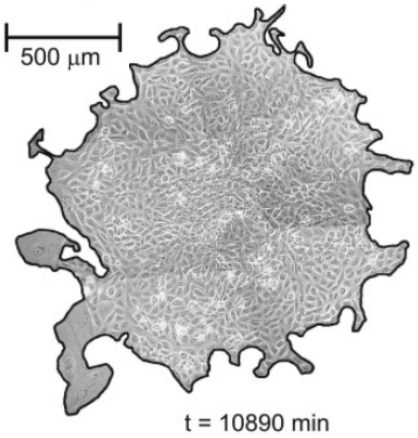
wild-type, hard agar (region B) ⇨ $\alpha = 0.78(2)$



mutant (surfactant -), soft agar (region D) ⇨ $\alpha = 0.50(1), \beta \approx 0.3$ KPZ !

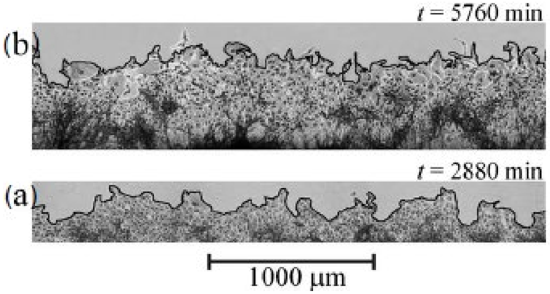


Cancer-like & Cancer Cell Growth



Experiments by Huergo et al. (2010-14) [9,37-39]

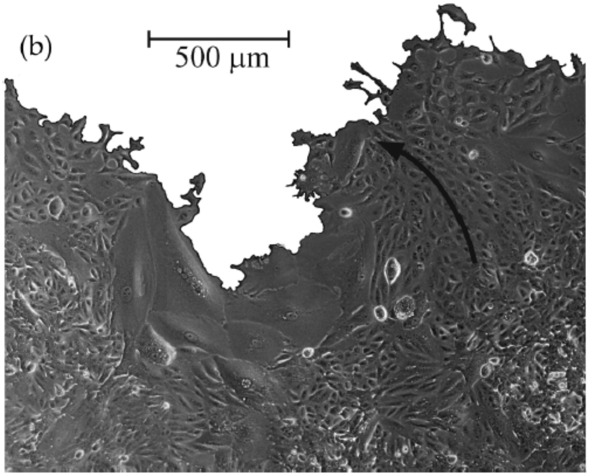
- Colony growth of cancer cells (HeLa) & cancer-like cells (Vero) on Petri dish
- Circular and flat geometries
- KPZ exponents were found consistently.
  e.g., Vero cell, flat case: $\alpha = 0.50(5),\ \beta = 0.33(2)$



- Earlier, Brú et al. [40,41] claimed the MH class (for conserved growth), but data also suggest KPZ exponents at longer scales.

- Exponents close to the quenched KPZ found under methylcellulose-containing medium.
  $\alpha = 0.63(4),\ \beta = 0.75(5)\ \ (\alpha_{\mathrm{qKPZ}} = \beta_{\mathrm{qKPZ}} = 0.63)$

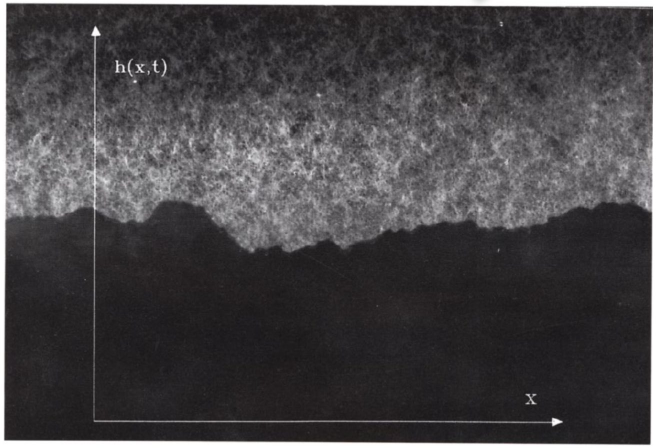
  Enlarged cells seem to serve as obstacles.



Paper Combustion

Slow combustion (or smoldering) of paper

Zhang et al. 1992 [42] ⇨ $\alpha = 0.71(5)$



- Lens cleaning paper with uniform KNO_3 (oxidization aid)
- No regulation of air flow
- Heat loss at boundary

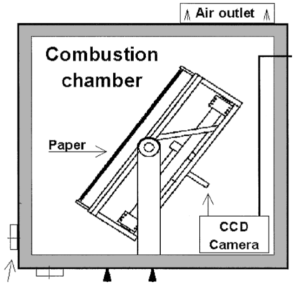
Maunuksela et al. 1997, Myllys et al. 2001 [2,3]

⇩

$\alpha = 0.48(1)$

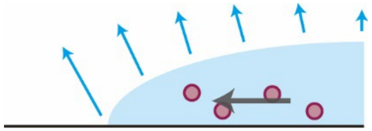
$\beta = 0.32(1)$

KPZ !



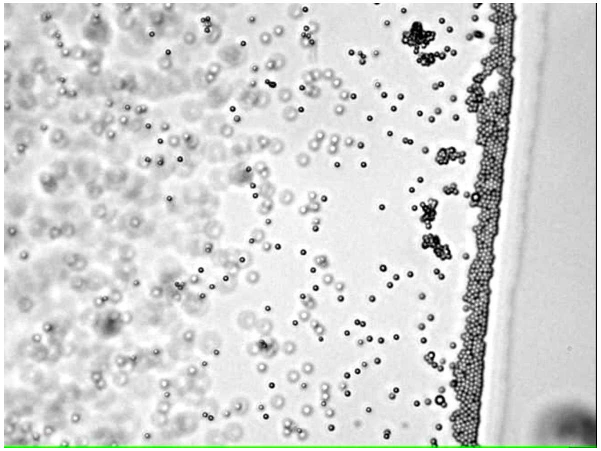
- 2 copier papers & lens paper with uniform KNO_3
- Regulated air flow
- Compensation of boundary heat loss
- Many properties of KPZ have been studied [2,3,43-46] (see [34] about distribution)

Coffee Ring (Particle Deposition)



Particle deposition in evaporating colloid (Yunker et al. 2013 [8])

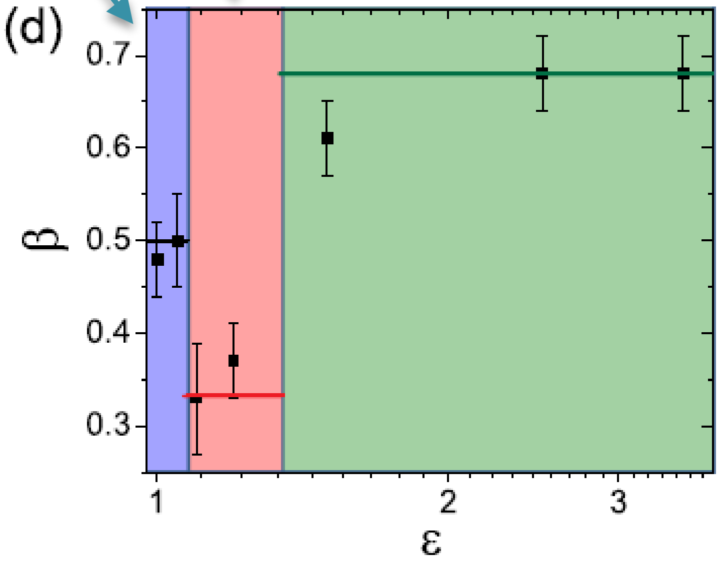
- Water droplet with polystyrene beads
- Controlled aspect ratio ε of beads.
  Larger ε → more deformed water surface
  → more long-ranged interaction
- Three regimes were found.



[7]

KPZ $(\alpha = \frac{1}{2},\ \beta = \frac{1}{3})$

uncorrelated deposition $(\alpha = 0,\ \beta = \frac{1}{2})$

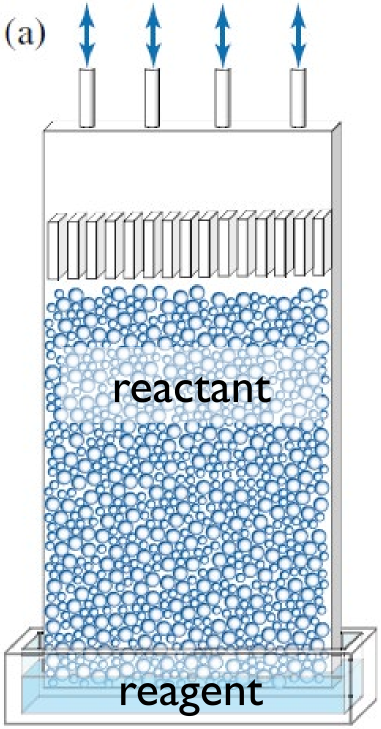


close to quench KPZ?
$(\alpha = \beta \approx 0.63)$
due to long-range effect?

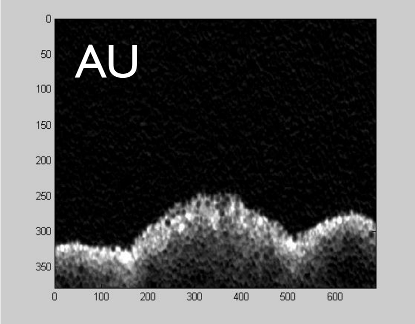
Chemical Reaction in Disordered Medium

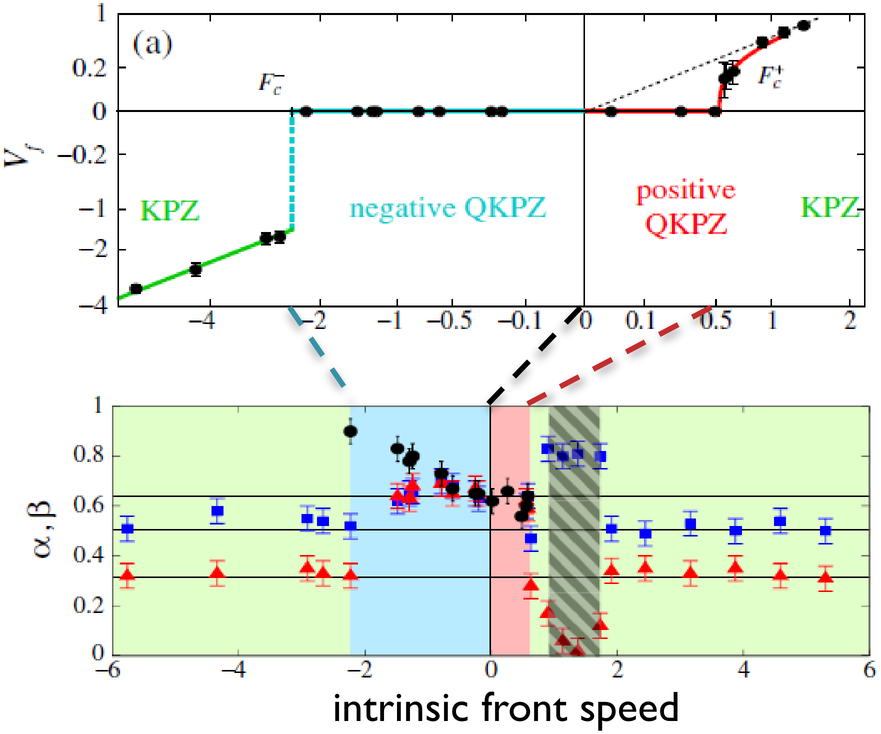
Chemical reaction fronts w/ advection & disorder (Atis et al. 2015 [47])

- $3H_3AsO_3 + IO_3^- + 5I^- \rightarrow 3H_3AsO_4 + 6I^-$
  in Hele-Shaw cell filled with bidisperse beads.
- Reaction propagates upward + external flow (up or down)
- Results:

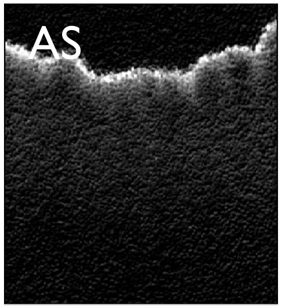


positive QKPZ





negative QKPZ



- Mapping to the quenched KPZ eq. is justified by eikonal approximation.

Chapter 5

Distribution and correlation properties
- stationary & non-stationary cases -

Main references [10]

5.0 Overview

See also reviews [10,14,16]

- Some models (including KPZ eq) in the 1D KPZ class turned out to be exactly solvable.
  - Totally asymmetric simple exclusion process (TASEP) (Johansson 2000 [48])
  - Polynuclear growth (PNG) model (Prähofer & Spohn 2000 [49])
  - Asymmetric simple exclusion process (ASEP) (Tracy & Widom 2009 [50])
  - KPZ equation (Sasamoto & Spohn 2010 [29,51], Amir et al. 2011 [52], Calabrese et al. 2010 [53], Dotsenko 2010 [54])

  and many more! (all related to integrability)

- Main consequences
  - δh's distribution & correlation functions were obtained.
  - Rich mathematical & theoretical structure (e.g., random matrix theory)
  - "Universality subclasses"
    different distribution & correlation laws
    depending on the initial condition / the global shape of interface.

5.1 PNG Model

- Time evolution

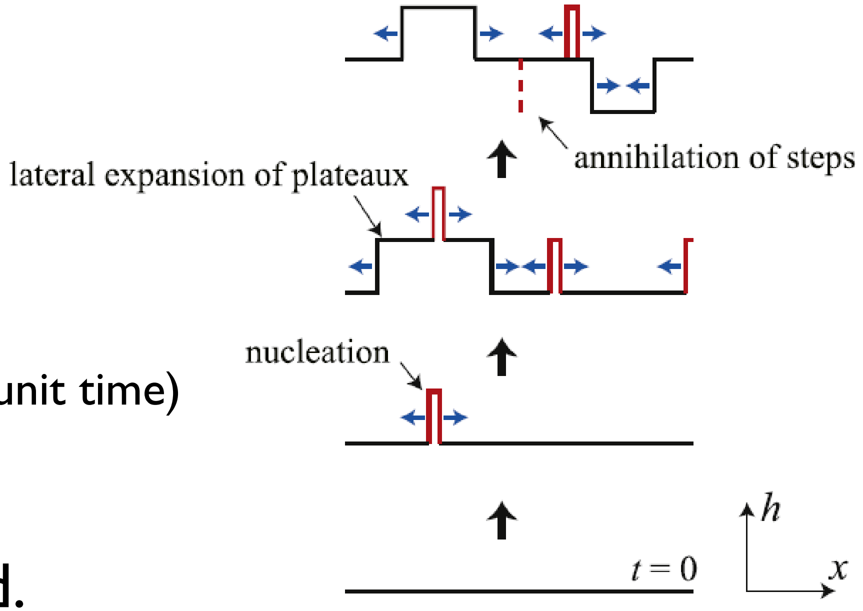
  (1) random local nucleation

  (at rate $\rho = 1$ per unit length, unit time)

  (2) deterministic lateral expansion

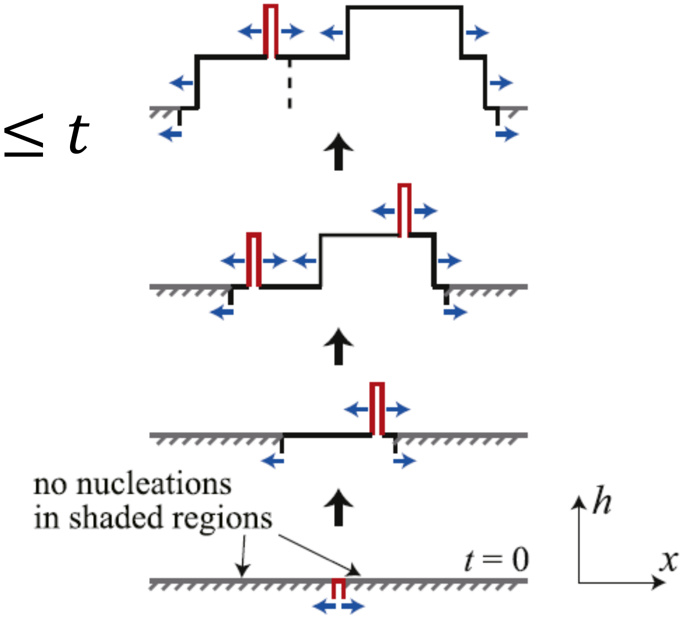
  (at speed $v = 1$)

  ➔ Flat interface is generated.



- Circular interface can also be made

  if nucleations occur only within $|x| \leq t$



Sketch from Hesse & Gross 2014

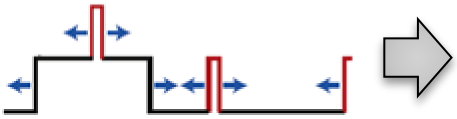
  - Mean height profile is indeed a semi-circle.

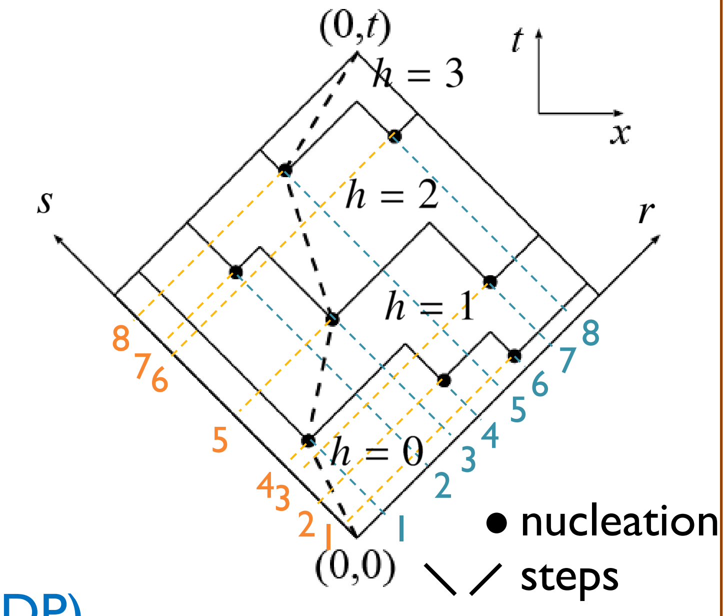
5.2 Circular PNG Interface

(Prähofer & Spohn 2000 [49])

Nucleations occur only within $|x| \le t$.

Let's draw a space-time plot!





$h(0,t)$

= # of lines to pass
when moving from $(0{,}0)$ to $(0,t)$

= max # of dots passed by directed polymer (DP)
btwn points $(0{,}0)$ & $(0,t)$
(point-to-point problem of DP)

random permutation

r: 1 2 3 4 5 6 7 8

s: $\underline{4}$ $\underline{7}$ 5 2 $\underline{8}$ 1 3 6

= length of longest increasing subsequences
in random permutations of Poisson-distributed length

= … (Young tableau, Robinson-Schensted correspondence) … (Baik et al., 1999 [55])

$$\xrightarrow{t\to\infty} 2\sqrt{\underline{S}} + S^{1/6}\chi_{\mathrm{GUE}} = \sqrt{2}t + \left(\frac{t}{\sqrt{2}}\right)^{1/3} \underline{\chi_{\mathrm{GUE}}}$$

◇'s area

random variable of "GUE Tracy-Widom distribution"

5.3 Tracy-Widom Distribution

(Tracy & Widom [56,57];
see also textbooks [58,59])

= distribution of the largest eigenvalue of Gaussian random matrices

e.g.) Gaussian Unitary Ensemble (GUE)

complex
Hermitian matrix $A = \begin{pmatrix} A_{11} & A_{12} & \dots & A_{1N} \\ A_{21} & A_{22} & \dots & A_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ A_{N1} & A_{N2} & \dots & A_{NN} \end{pmatrix}$

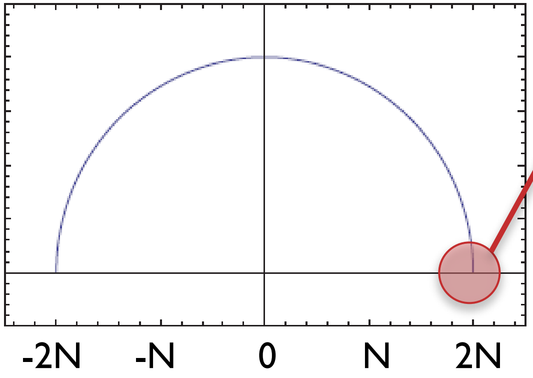
Gaussian
mean 0 variance *N*/2

$$A_{ij} = \bar{A}_{ji} = a_{ij} + ib_{ij}$$

$$A_{ii} = a_{ii}$$

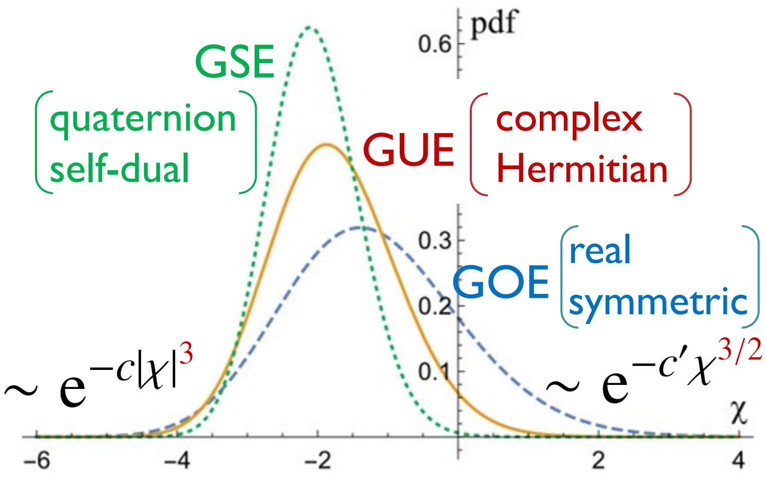
mean 0 variance *N*

distribution of all *N* eigenvalues
(Wigner's semicircle law)



$$\boxed{\lambda_{\max} \simeq 2N + N^{1/3}\chi_{\mathrm{GUE}}}$$

$\mathrm{pdf}(\chi_{\mathrm{GUE}}) \equiv$
GUE Tracy-Widom dist.



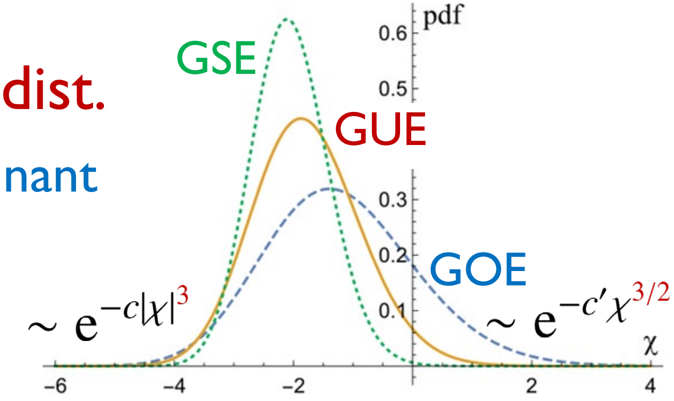
$\sim \mathrm{e}^{-c|\chi|^3}$

$\sim \mathrm{e}^{-c'\chi^{3/2}}$

5.3 Tracy-Widom Distribution

(Tracy & Widom [56,57]; see also textbooks [58,59])

Some remarks on GUE Tracy-Widom dist.



- Analytic expression using Fredholm determinant

$$\boxed{\mathrm{Prob}(\chi_{\mathrm{GUE}} \leq s) = \det(1 - P_s K_{\mathrm{Ai}} P_s)}$$

P_s: projection onto $[s, \infty)$

$K_{\mathrm{Ai}}(x,y) \equiv \int_0^\infty d\lambda\, \mathrm{Ai}(x+\lambda)\mathrm{Ai}(y+\lambda)$: Airy kernel

det: Fredholm determinant, $\det(1+zK) \equiv \sum_{n=0}^{\infty} \frac{z^n}{n!} \int_{(-\infty,\infty)^n} \det[K(x_i, x_j)_{i,j=1}^n] \mathrm{d}x_1 \cdots \mathrm{d}x_n$

(see [60,61] for numerical evaluation of Fredholm determinant)

- Another expression using Painlevé II equation [62]

$$\boxed{\frac{d^2u}{dx^2} = 2u(x)^3 + xu(x)}$$

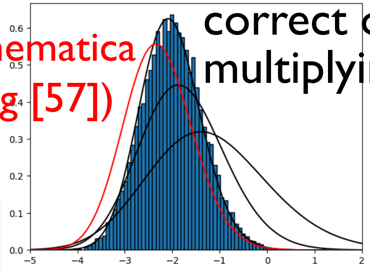
with its global positive solution $u(x)$ and $g(x)$ s.t. $g''(x) = u(x)^2, g(x) \xrightarrow{x\to\infty} 0$,

$$\boxed{\mathrm{Prob}(\chi_{\mathrm{GUE}} \leq s) = e^{-g(s)}}$$

- For users:
  - ➢ Prähofer & Spohn's numerical table [63]
  - ➢ Mathematica

TracyWidomDistribution [β]

represents a Tracy–Widom distribution with Dyson index β.



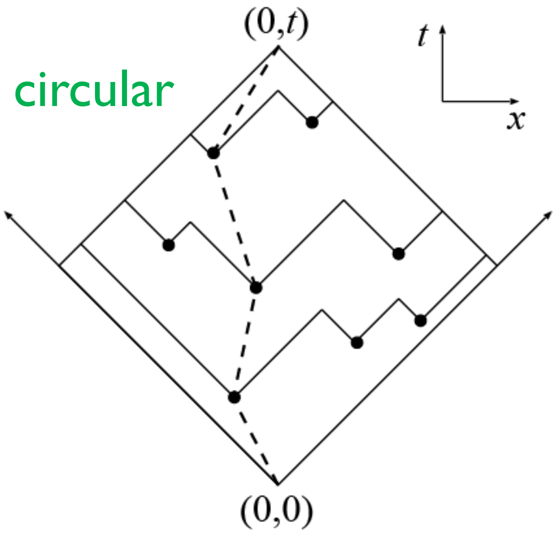
Mathematica (using [57])

correct one, multiplying χ by $2^{-1/6}$

※ Mathematica's GSE TW seems to be wrong (thx to Y. Ito on this)

(Prähofer & Spohn 2000 [49])

5.4 Flat PNG Interface



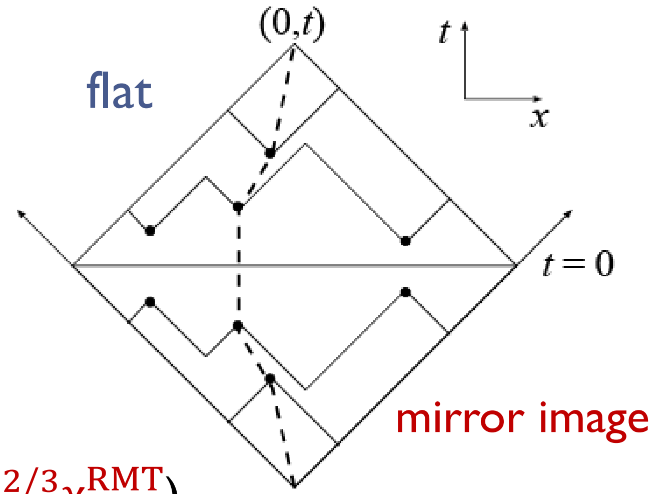
PNG circular interfaces

Nucleations restricted to $|x| \leq t$

⇩

Consider a square set by $(0{,}0)$ & $(0, t)$

"point-to-point directed polymer" ⇨ GUE



PNG flat interfaces

No constraint on nucleations

⇩

Consider a triangle set by $t = 0$ & $(0, t)$

"line-to-point directed polymer"

⇩ mirror image

Equivalent to square / point-to-point problem,

but with time-reversal symmetry ⇨ GOE

(precisely, $\chi_{\mathrm{GOE}}^{\mathrm{KPZ}} = 2^{-2/3} \chi_{\mathrm{GOE}}^{\mathrm{RMT}}$)

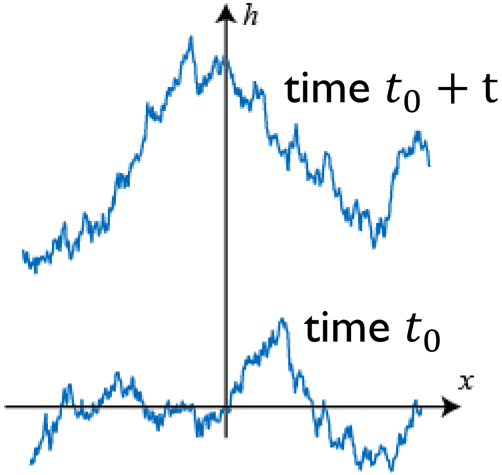
Different geometries (or initial conditions) lead to different symmetries.

KPZ class splits into a few "universality subclasses.

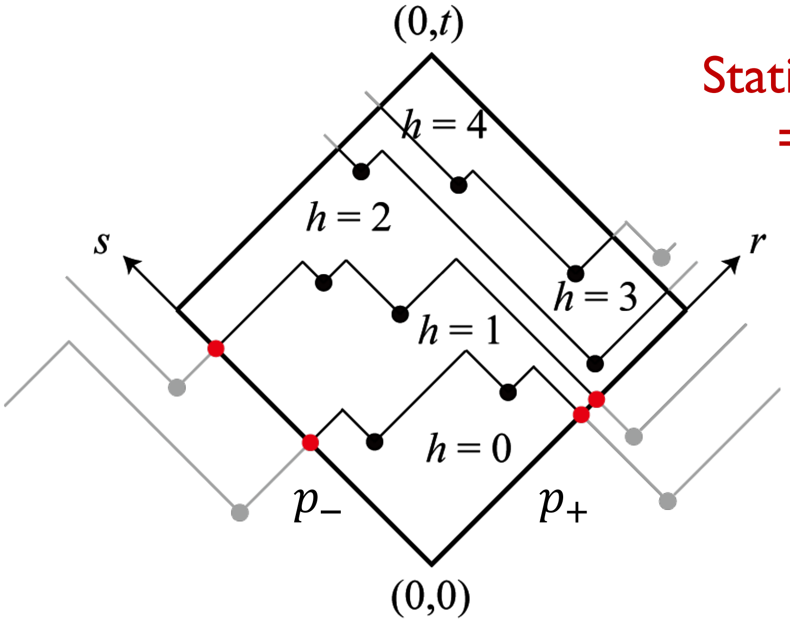
(Prähofer & Spohn 2000 [49])

5.5 Stationary PNG Interface

- Quantity of interest:
  height difference
  between two times in the stationary state
  $h(0, t_0 + t) - h(0, t_0) = h(0, t)$
  ↑by setting $t_0 = 0, h(0,0) = 0$

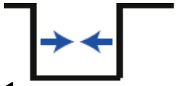


- In PNG space-time plot



Stationary PNG problem
= circular PNG + extra nucl's on r, s axes

Boundary nucleation rate $p_\pm$?

- $p_\pm = \sqrt{2} \times$ (step density, $\rho_{\lrcorner} = \rho_{\llcorner}$)
- Nucleation-annihilation balance
  $\rho \mathrm{d}x\mathrm{d}t = (\rho_{\lrcorner}\mathrm{d}x)(\rho_{\llcorner} 2v\mathrm{d}t)$
  
  → $\rho_{\lrcorner} = \rho_{\llcorner} = 1/\sqrt{2}$, $p_\pm = 1$

(Prähofer & Spohn 2000 [49])

5.5 Stationary PNG Interface

Stationary PNG = Circular PNG + Boundary nucleations

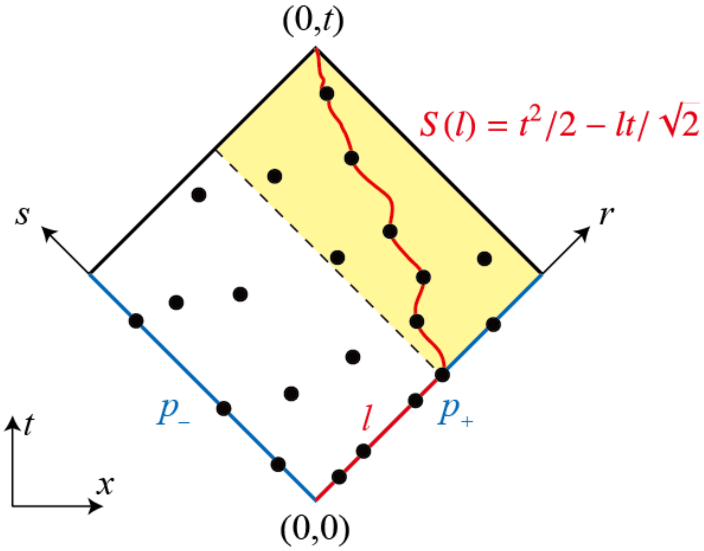
$$\boxed{h = \max_{l,\pm}[h_{\text{bulk}}(l) + h_\pm(l)]}$$

with $h_{\text{bulk}}(l) \simeq 2\sqrt{S(l)} + S(l)^{1/6}\chi_{\text{GUE}}$,

$$h_\pm(l) \simeq (p_\pm l) + (p_\pm l)^{1/2}\chi_{\text{Gauss}}$$

l is determined by the leading terms:

$$l_{\max} = \operatorname{argmax}\left[2\sqrt{S(l)} + p_\pm l\right] = \frac{t}{\sqrt{2}}\left(1 - \frac{1}{p_\pm^2}\right)$$



- $p_\pm < 1$ ➜ bulk dominant, GUE Tracy-Widom
- p_+ or $p_- > 1$ ➜ boundary dominant, Gaussian (if $p_+ \neq p_-$)
- $p_\pm = 1$ ➜ critical, Baik-Rains distribution [50]

⇦ dist. for stationary PNG!

Baik-Rains distribution [64,49]

- No known link to random matrix.
- Definition using Painlevé II: $\downarrow f(x)$ s.t. $f'(x) = -u(x)$ and $f(x) \xrightarrow{x\to\infty} 0$; see p.40 for g

$$\boxed{\mathrm{Prob}[\chi_{\mathrm{BR}} \le s] = [1 - (s + 2f''(s) + 2g''(s))g'(s)]e^{-2f(s)-g(s)}}$$

5.6 Universality Subclasses

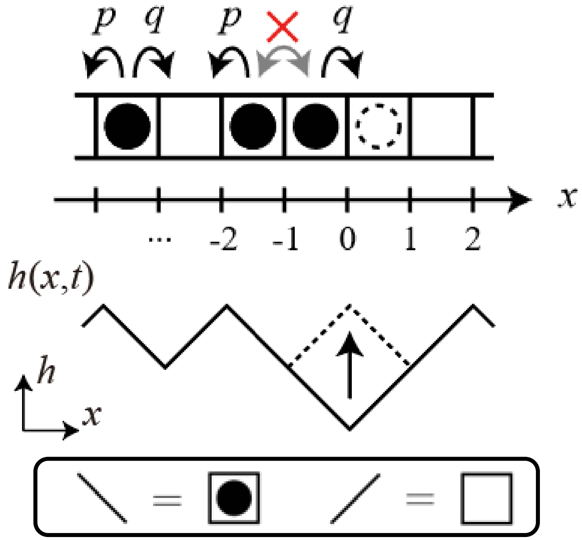
	Circular	**Flat**	**Stationary**
exponents	$\alpha = 1/2\,, \beta = 1/3\,, z = 3/2$ (common for all subclasses)		
distribution	GUE Tracy-Widom	GOE Tracy-Widom	Baik-Rains

These are believed to be universal in the 1D KPZ class though mathematical evidence can only be obtained for integrable models.

Let's see how they appear in other (integrable) models.

5.7 Asymmetric Simple Exclusion Process (ASEP)

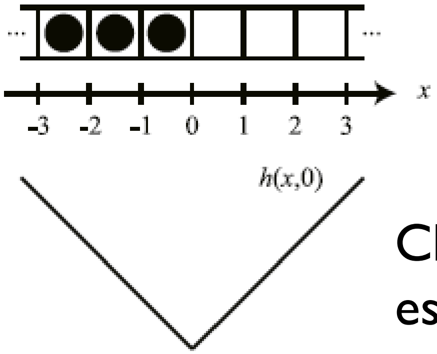
- Lattice model of stochastic particle transport
  - Asymmetric hopping rates: $p < q$ (particles hop preferentially to right)
  - Volume exclusion (no particle overlap)
- Integrable model (see reviews [65,66])
- Mapped to interface growth model. KPZ universality class.



The three universality subclasses

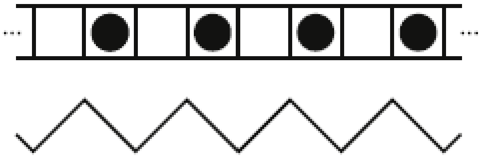
circular:

step initial condition



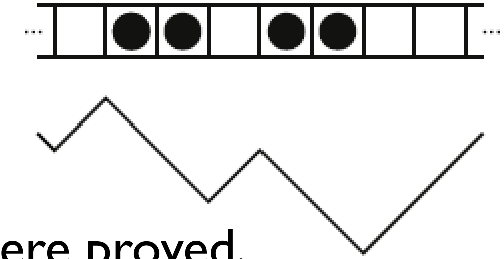
flat:

alternating IC



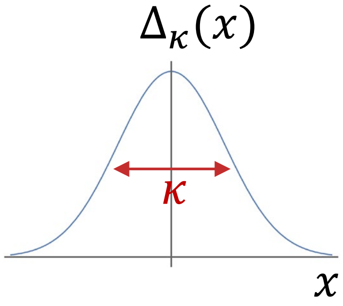
stationary:

Bernoulli IC



Characteristic distributions were proved, especially for totally ASEP ($p = 0$) [14,16]

5.8 KPZ Equation (see, e.g., review [10])



1D KPZ equation (coefficients fixed, w/o loss of generality)

$$\boxed{\frac{\partial h}{\partial t} = \frac{1}{2}\frac{\partial^2 h}{\partial x^2} + \frac{1}{2}\left(\frac{\partial h}{\partial x}\right)^2 + \eta_\kappa}$$

$\eta_\kappa(x,t)$: spatially correlated noise

$\langle \eta_\kappa(x,t)\rangle = 0, \langle \eta_\kappa(x,t)\eta_\kappa(x',t')\rangle = D\Delta_\kappa(x-x')\delta(t-t')$

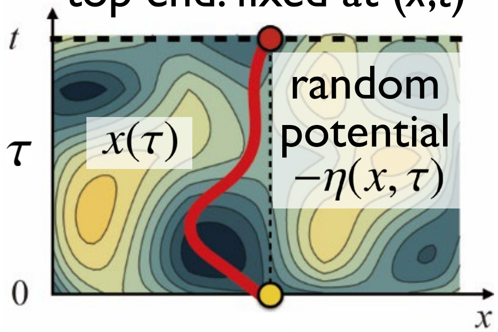
Cole-Hopf transformation $\boxed{Z(x,t) = e^{h(x,t)}}$

stochastic heat equation

$$\boxed{\frac{\partial Z}{\partial t} = \frac{1}{2}\frac{\partial^2 Z}{\partial x^2} + \eta_\kappa Z} = \frac{1}{2}\frac{\partial^2 Z}{\partial x^2} + \eta_\kappa \circ Z - \frac{1}{2}\Delta_\kappa(0)Z$$

path integral (Feynman-Kac formula)

top end: fixed at (*x*,*t*)



bottom end: dist'd by $Z(x,0)$

directed polymer (DP)'s partition function

$$\boxed{Z(x,t) = \int \mathcal{D}x(\tau)\ \exp\left\{-\int \mathrm{d}\tau \left[\underbrace{\frac{1}{2}\left(\frac{\mathrm{d}x}{\mathrm{d}\tau}\right)^2}_{\text{polymer elasticity}} \underbrace{-\ \eta_\kappa(x,\tau)}_{\text{random potential}} + \frac{1}{2}\Delta_\kappa(0)\right]\right\}}$$

$h(x,t) = \log Z(x,t)$
= -(DP free energy)

5.8 KPZ Equation (see, e.g., review [10])

directed polymer (DP)'s partition function

$$\boxed{Z(x,t) = \int \mathcal{D}x(\tau)\ \exp\left\{-\int \mathrm{d}\tau \left[\frac{1}{2}\left(\frac{\mathrm{d}x}{\mathrm{d}\tau}\right)^2 - \eta_\kappa(x,\tau) + \frac{1}{2}\Delta_\kappa(0)\right]\right\}}$$

Let's consider Nth-order moment $\Psi(\vec{x},t) \equiv \langle Z(x_1,t)\cdots Z(x_N,t)\rangle$

with $\eta_n \equiv \eta_\kappa(x_n(\tau),\tau)$, $\left\langle \mathrm{e}^{\sum_n \eta_n \mathrm{d}\tau}\right\rangle = \int \left(\prod_n \mathrm{d}\eta_n\right) \mathrm{e}^{\sum_n \eta_n \mathrm{d}\tau} \exp\left(\sum_{n.m} \frac{\eta_n\eta_m}{2\Delta_\kappa(x_n - x_m)/\mathrm{d}\tau}\right)$

$$\propto \exp\left(\frac{1}{2}\sum_{n,m} \Delta_\kappa(x_n - x_m)\mathrm{d}\tau\right)$$

← Gauss integral over η_n

$$\boxed{\begin{aligned}\Psi(\vec{x},t) &\equiv \langle Z(x_1,t)\cdots Z(x_N,t)\rangle \\ &= \int \mathcal{D}x_1(\tau)\cdots\mathcal{D}x_N(\tau)\ \exp\left\{-\int \mathrm{d}\tau\left[\frac{1}{2}\sum_n\left(\frac{\mathrm{d}x_n}{\mathrm{d}\tau}\right)^2 - \frac{1}{2}\sum_{n\neq m}\Delta_\kappa(x_n - x_m)\right]\right\}\end{aligned}}$$

applying the Feynman-Kac formula in the other way

$$\boxed{\frac{\partial \Psi}{\partial t} = -\hat{H}_\kappa \Psi \quad \text{with} \quad \hat{H}_\kappa = -\frac{1}{2}\sum_n \frac{\partial^2}{\partial {x_n}^2} - \frac{1}{2}\sum_{n\neq m}\Delta_\kappa(x_n - x_m)}$$

white noise limit $\kappa \to 0$

$$\boxed{\hat{H}_{\mathrm{LL}} = -\frac{1}{2}\sum_n \frac{\partial^2}{\partial {x_n}^2} - \frac{1}{2}\sum_{n\neq m}\delta(x_n - x_m)}$$

attractive Lieb-Liniger model
(quantum integrable model of N bosons)

5.8 KPZ Equation (Recapped)

(see, e.g., review [10])

1D KPZ equation (coefficients fixed, w/o loss of generality)

$$\boxed{\frac{\partial h}{\partial t} = \frac{1}{2}\frac{\partial^2 h}{\partial x^2} + \frac{1}{2}\left(\frac{\partial h}{\partial x}\right)^2 + \eta}$$

$\eta(x,t)$: white Gaussian noise

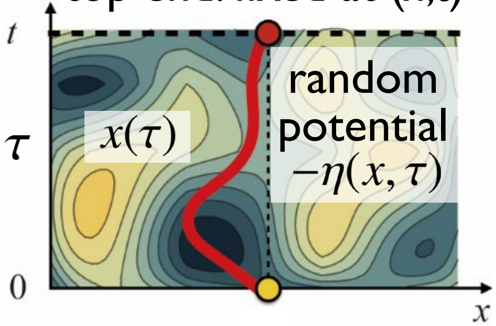
Cole-Hopf transformation $\boxed{Z(x,t) = \mathrm{e}^{h(x,t)}}$

stochastic heat equation

$$\boxed{\frac{\partial Z}{\partial t} = \frac{1}{2}\frac{\partial^2 Z}{\partial x^2} + \eta Z}$$

path integral (Feynman-Kac formula)

top end: fixed at (*x*,*t*)



bottom end: dist'd by $Z(x,0)$

directed polymer (DP)'s partition function

$$\boxed{Z(x,t) = \int \mathcal{D}x(\tau)\ \exp\left\{-\int \mathrm{d}\tau \left[\underbrace{\frac{1}{2}\left(\frac{\mathrm{d}x}{\mathrm{d}\tau}\right)^2}_{\text{polymer elasticity}} \underbrace{- \eta(x,\tau)}_{\text{random potential}}\right]\right\}}$$

$h(x,t) = \log Z(x,t)$

height = -(DP free energy)

for Nth-order moment $\Psi(\vec{x},t) \equiv \langle Z(x_1,t)\cdots Z(x_N,t)\rangle$

N-body bosons (attractive Lieb-Liniger model, quantum integrable model)

$$\boxed{\frac{\partial \Psi}{\partial t} = -H_{\mathrm{LL}}\Psi, \quad H_{\mathrm{LL}} \equiv -\frac{1}{2}\sum_n \frac{\partial^2}{\partial {x_n}^2} - \frac{1}{2}\sum_{n \neq m}\delta(x_n - x_m)}$$

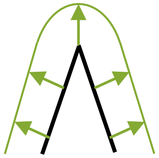
5.8 KPZ Equation

- Recap: $\boxed{\text{interface } h(x,t)} \Leftrightarrow \boxed{\text{polymer part. func. } Z(x,t) = e^{h}} \Leftrightarrow \boxed{\text{bosons}}$
- Initial conditions?

circular case

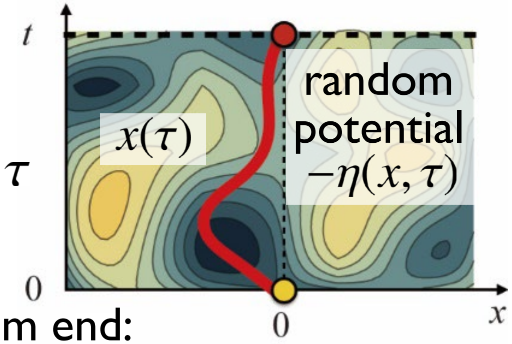
$$h(x,0) = -\kappa|x|$$

$$Z(x,0) = e^{h(x,0)} \xrightarrow{\kappa\to\infty} \delta(x)$$



polymer picture

top end: fixed at a point



bottom end:
dist'd by $Z(x,0) = \delta(x)$ → fixed at (0,0)

circular = "point-to-point problem"

5.8 KPZ Equation

Calabrese et al. 2010 [53]
Dotsenko 2010 [54]

KPZ eq ⇨ attractive Lieb-Liniger model

$$\boxed{H_{\mathrm{LL}} = -\frac{1}{2}\sum_{n}\frac{\partial^2}{\partial {x_n}^2} - \frac{1}{2}\sum_{n\neq n'}\delta(x_n - x_{n'})}$$

initial condition for circular case

We need $\boxed{\langle Z(0,t)^N\rangle = \langle\vec{0}\,|\,\mathrm{e}^{-H_{\mathrm{LL}}t}\,|\,\vec{0}\rangle = \sum_{\mu:\text{ eigenstate}} \mathrm{e}^{-E_\mu t}|\langle\vec{0}\,|\psi_\mu\rangle|^2}$ $= \sum_{J=1}^{N}\frac{1}{J!}\prod_{j=1}^{J}\left(\sum_{M_j}\int \mathrm{d}k_j\right)\mathrm{e}^{-E_\mu t}|\langle\vec{0}\,|\psi_\mu\rangle|^2$

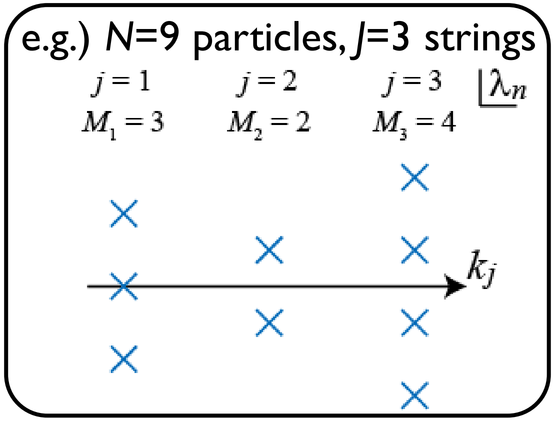
Bethe Ansatz: $\boxed{\psi_\mu(x_1,\cdots,x_N) \propto \sum_{P} A_P \exp\left(\mathrm{i}\sum_{n=1}^{N}\lambda_{P(n)}x_n\right)}$

P : permutation
λ_n : (quasi)momentum

For the attractive LL model: $A_P = (-1)^P\prod_{n<n'}[\lambda_{P(n)} - \lambda_{P(n')} + \mathrm{i}\,\mathrm{sgn}(x_n - x_{n'})]$

Particles form "strings" (jth string has M_j particles)

$$\boxed{\lambda_n = k_j + \frac{\mathrm{i}}{2}(M_j + 1 - 2m_j) \quad (m_j = 1, 2, \cdots, M_j)}$$


e.g.) N=9 particles, J=3 strings

$$E_\mu = \frac{1}{2}\sum_{n=1}^{N}\lambda_n^2 = \frac{1}{2}\sum_{j=1}^{J}M_jk_j^2 - \frac{1}{24}\sum_{j=1}^{J}M_j^3 + \frac{N}{24}$$

diverging, but…

$$\exp(n^3/3) \approx \int_{-\infty}^{\infty}\mathrm{d}y\,\mathrm{Ai}(y)\mathrm{e}^{ny}$$

Airy kernel of Tracy-Widom dist.
$K(x,y) = \int_0^\infty \mathrm{d}\lambda\,\mathrm{Ai}(x+\lambda)\,\mathrm{Ai}(y+\lambda)$

※ So this is non-rigorous. [29,51,52] are rigorous.

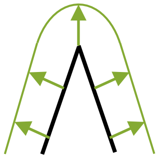
5.8 KPZ Equation

- Recap: $\boxed{\text{interface } h(x,t)} \Leftrightarrow \boxed{\text{polymer part. func. } Z(x,t) = e^{h}} \Leftrightarrow \boxed{\text{bosons}}$
- Universality subclasses?

circular case

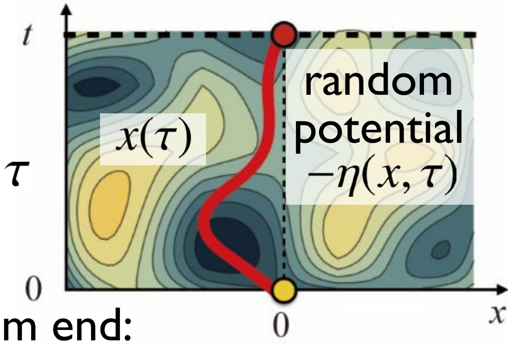
$$h(x,0) = -\kappa|x|$$

$$Z(x,0) = e^{h(x,0)} \xrightarrow{\kappa\to\infty} \delta(x)$$



polymer picture

top end: fixed at a point



bottom end:
dist'd by $Z(x,0) = \delta(0) \to$ fixed at (0,0)

circular = "point-to-point problem"
(→ GUE Tracy-Widom [53,54])

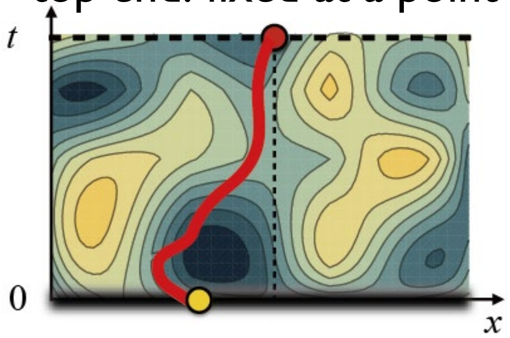
flat case

$$h(x,0) = 0$$

$$Z(x,0) = e^{h(x,0)} = \text{const}$$

polymer picture

top end: fixed at a point



bottom end: uniformly distributed

flat = "line-to-point problem"
(→ GOE Tracy-Widom [67])

5.9 Height Rescaling [10,13,68,69]

	Circular	Flat	Stationary
exponents	$\alpha = 1/2\,, \beta = 1/3\,, z = 3/2$ (common for all subclasses)		
distribution	GUE Tracy-Widom	GOE Tracy-Widom	Baik-Rains

How to compare $h(x,t)$ & universal distribution variable χ?

- Growth law: $\boxed{h(x,t) \simeq v_\infty t + (\Gamma t)^{1/3} \chi\left(\frac{x}{\xi(t)}\right)}$ with corr. length $\xi(t)$
- Stationary Brownian profile: $\boxed{C_h(\ell,t) \equiv \left\langle [h(x+\ell,t) - h(x,t)]^2 \right\rangle \simeq A\ell}$
- KPZ nonlinearity: $\boxed{\lambda = \lim_{u\to 0} \frac{d^2}{du^2} v_\infty(u)}$ with mean slope $u = \langle \nabla h \rangle$

  For isotropic growth, we have $v_\infty(u) = \sqrt{1+u^2} v_\infty$ $\therefore \boxed{\lambda = v_\infty}$ (isotropic)

Then, $\boxed{\Gamma = \frac{1}{2} A^2 \lambda,\ \ \xi(t) = \frac{2}{A} (\Gamma t)^{-2/3}}$ [10,69]

5.10 Correlation Properties

What about correlation properties?

	Circular	Flat	Stationary
exponents	$\alpha = 1/2\,, \beta = 1/3\,, z = 3/2$ (common for all subclasses)		
distribution	GUE Tracy-Widom	GOE Tracy-Widom	Baik-Rains
Limiting process for spatial profile	Airy$_2$ process [70,71] $\mathcal{A}_2(\tau)$	Airy$_1$ process [72] $\mathcal{A}_1(\tau)$	Brownian motion $B(\tau)$

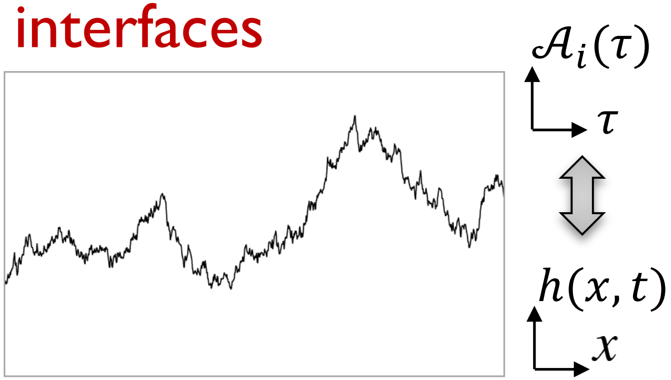
Limiting stochastic process, "Airy process" $\mathcal{A}_i(t)$ (review [73]), describe spatial profiles of circular/flat interfaces

rescaled height

$$\boxed{\chi\left(\frac{x}{\xi(t)}\right) \to \begin{cases} \mathcal{A}_2(\tau) - \tau^2 & \text{(circ.)} \\ \mathcal{A}_1(\tau) & \text{(flat)} \end{cases}}$$

with $\tau \equiv x/\xi(t)$, $\xi(t)$ corr. length

※convention for Airy$_1$ may differ [10].



Determinantal formulae obtained [70-73].

∴ Spatial correlation properties are known.

e.g., n-point correlation $\langle h(x_1,t)h(x_2,t)\cdots h(x_n,t)\rangle$

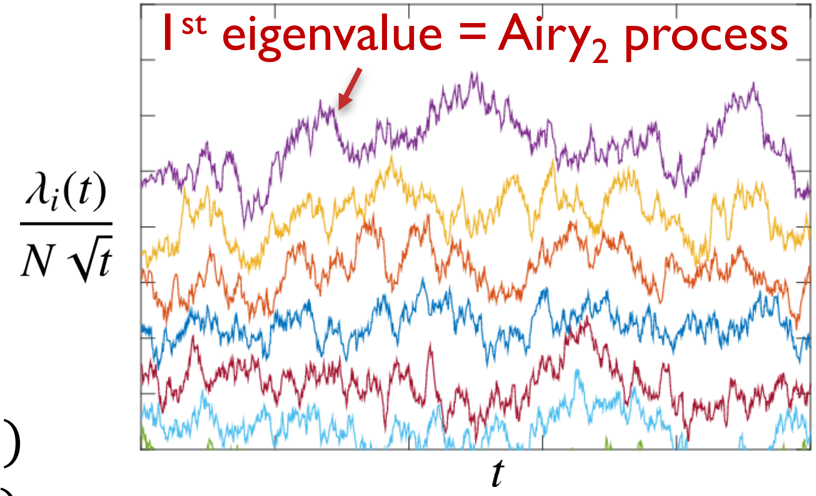
5.10.1 Airy$_2$ Process for Circular Case [70,71,73]

Airy$_2$ process
= Top eigenvalue of Dyson's Brownian motion for GUE matrices

$$A(t) = \begin{pmatrix} A_{11}(t) & \cdots & A_{1N}(t) \\ \vdots & \ddots & \vdots \\ A_{N1}(t) & \cdots & A_{NN}(t) \end{pmatrix}$$

eigenvalues: $\lambda_i(t)$

- Each $A_{ij}(t) = a_{ij}(t) + \mathrm{i}b_{ij}(t)$ does independent Brownian motion.
- Then, $\lambda_i(t) \sim N\sqrt{t}$



Alternatively, consider
Ornstein-Uhlenbeck process

$$\frac{dA}{dt} = -A(t) + \Xi(t)$$

$$\langle \Xi_{ij}(t)\Xi_{ij}(t) \rangle = \begin{cases} 2N\delta(t-t') & (i=j) \\ N\delta(t-t') & (i \neq j) \end{cases}$$

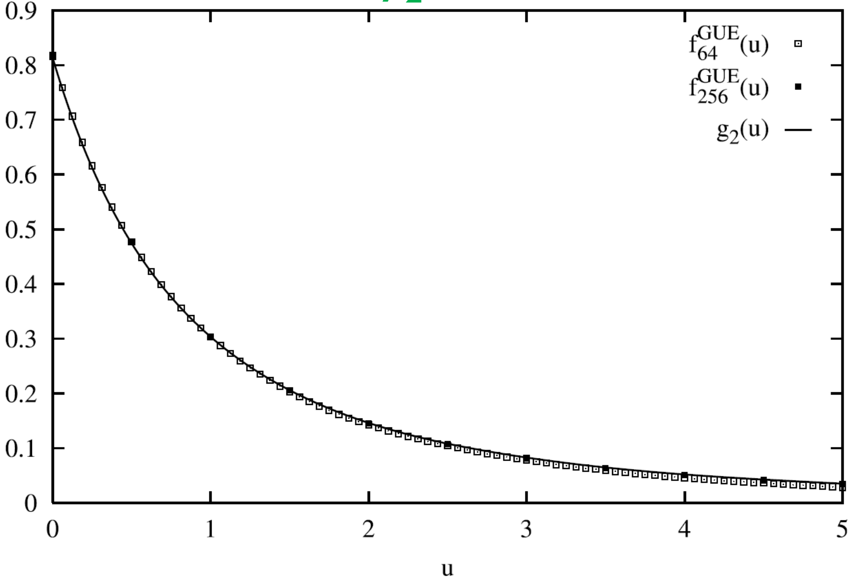
Then, $\boxed{N^{-1/3}\left[\lambda_{\max}\left(\frac{\tau}{N^{1/3}}\right) - 2N\right] \to \mathcal{A}_2(\tau)}$

5.10.2 Airy$_\mathrm{I}$ Process for Flat Case

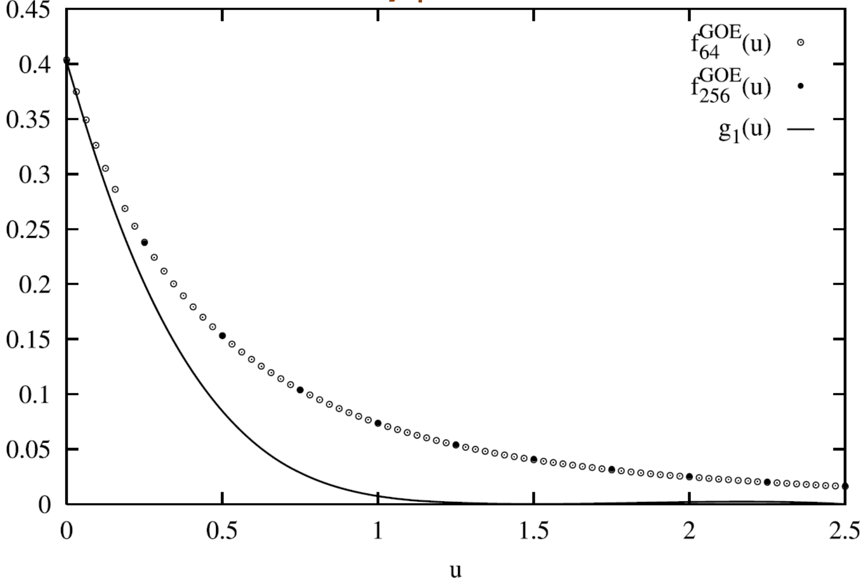
Airy$_\mathrm{I}$ process
= Top eigenvalue of Dyson's Brownian motion for GOE? No!

Numerical evaluation of determinantal formulae
for the 2-point function $g_i(u) \equiv \langle \mathcal{A}_i(\tau+u)\mathcal{A}_i(\tau)\rangle$ [74]

Airy$_2$ vs GUE



Airy$_\mathrm{I}$ vs GOE



Theoretically, $g_2(u) \sim u^{-2}$ [74,75]
power-law decay

$g_1(u) \sim \exp\left(-\frac{1}{3}u^3\right)$ [76]
super-exponential decay!

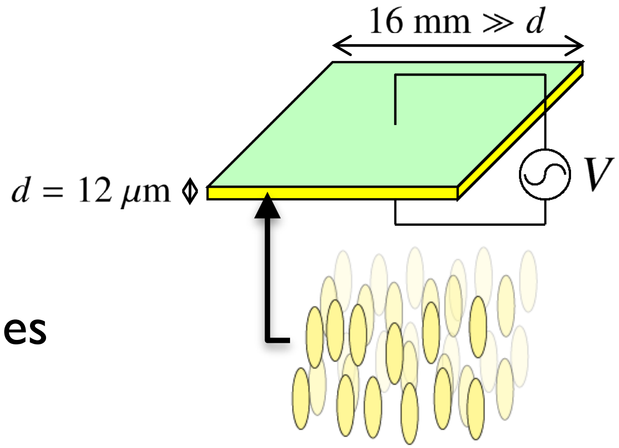
Chapter 6

Experimental test of distribution and correlation properties

Main references [6,10]

Reminder:
Universality in distribution & correlation properties was checked only for integrable models.
What about non-integrable systems?
Are these robust enough to arise in real phenomena?

Liquid-Crystal Experiment



liquid crystal MBBA

Convection of nematic liquid crystal

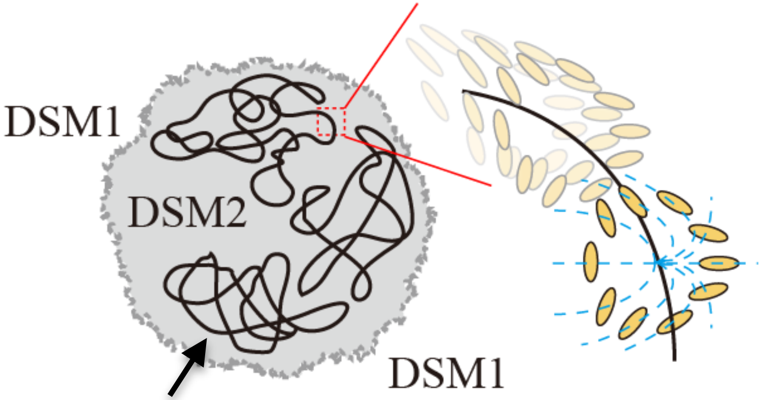
driven by electric field

thanks to nematic anisotropy of electric properties

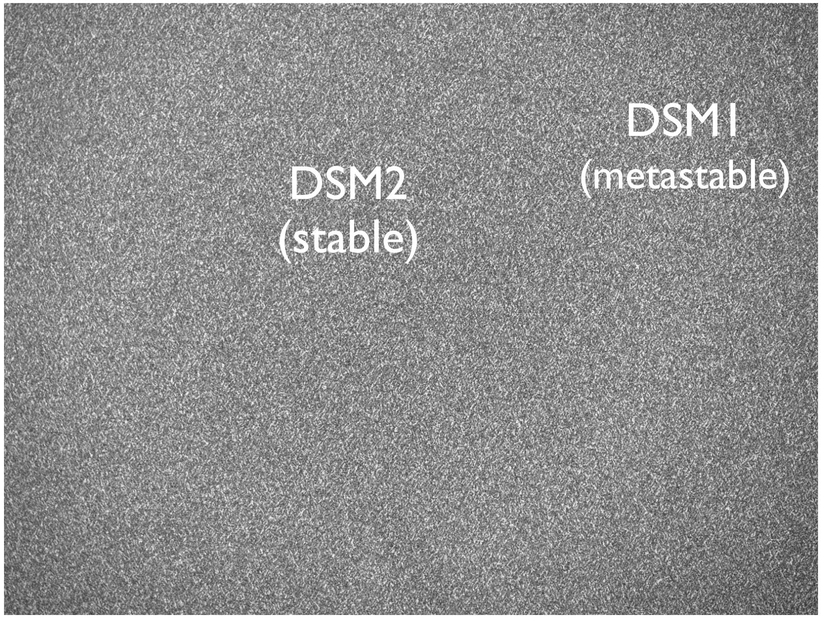
Two turbulent states at high enough V

Metastable: DSM1 = defect-less turbulence

Stable: DSM2 = defect-filled turbulence



Topological defect lines
in nematic director field

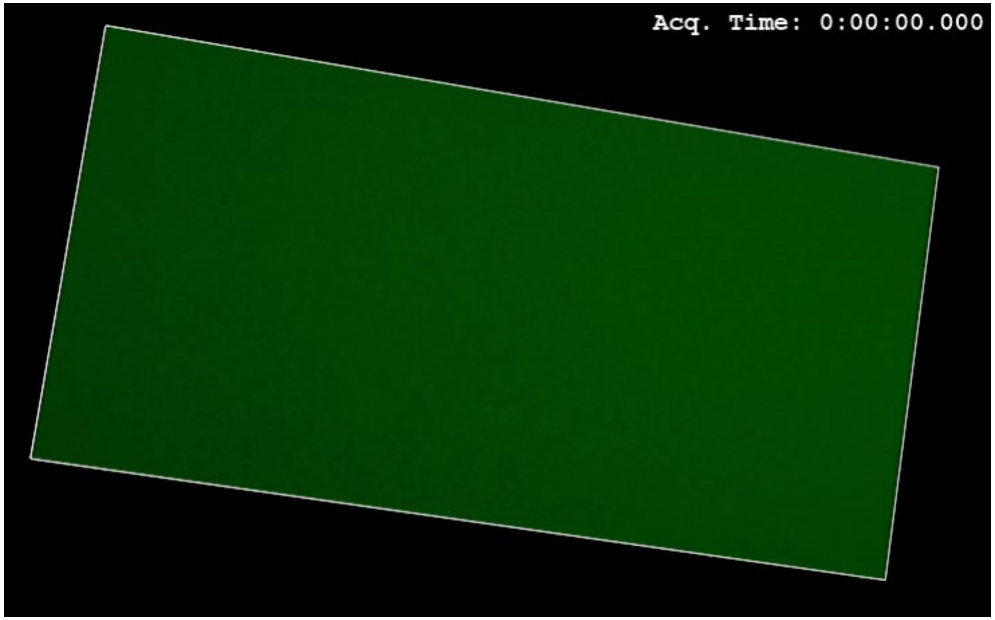


35V, 250Hz Speed x2, — 200 μm
(homeotropic alignment)

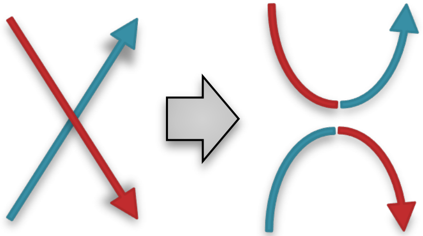
Growing DSM2 interfaces!

DSM2 = Topological Defect Turbulence

Direct visualization of defect lines in relaxation from DSM2

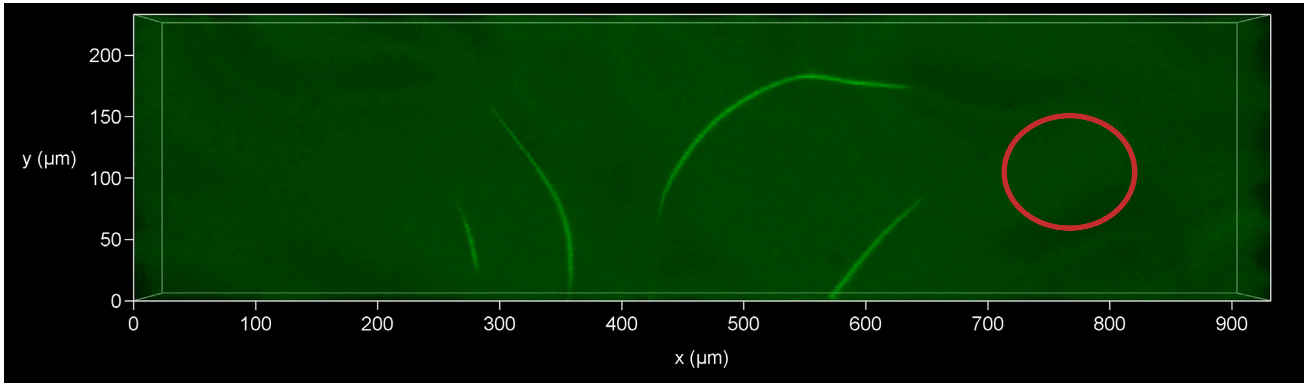


(Zushi & Takeuchi, PNAS 2022 [77]
arXiv 2024 [78])



reconnections of defect lines

and in DSM2 turbulence!

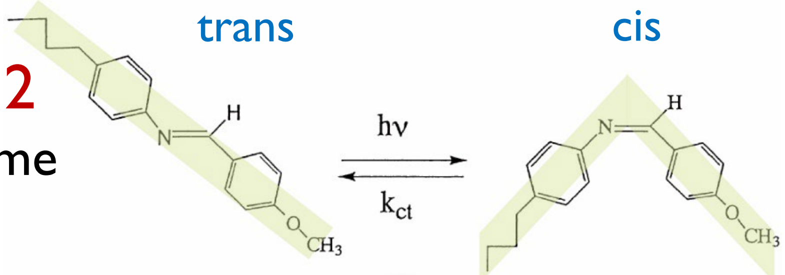


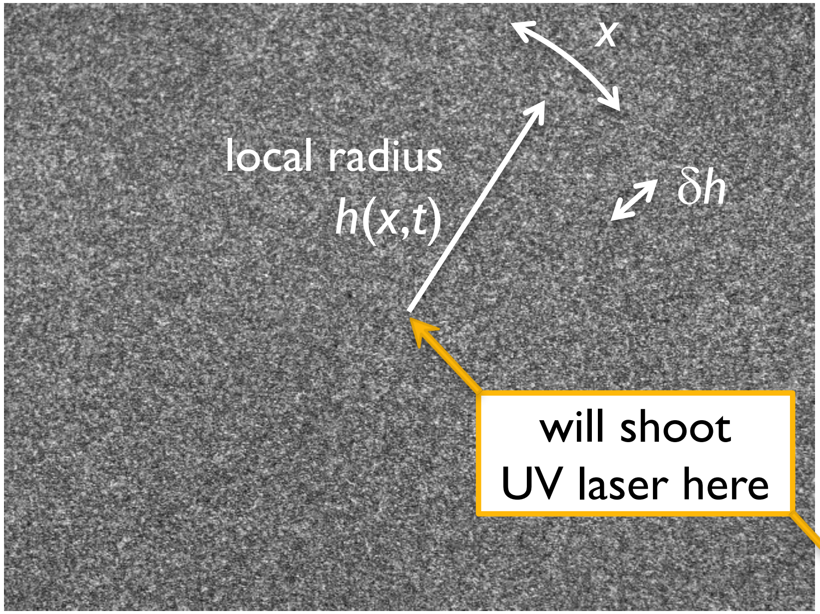
Trigger the Growth

Used UV laser to generate DSM2

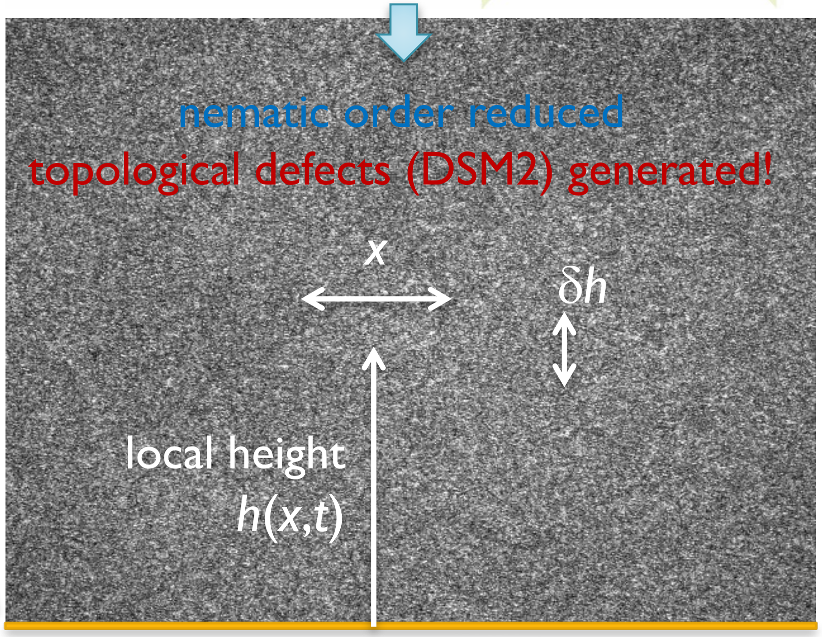
- Growth starts at target position & time
- Can design the initial shape!

photo-isomerization of MBBA





will shoot UV laser here

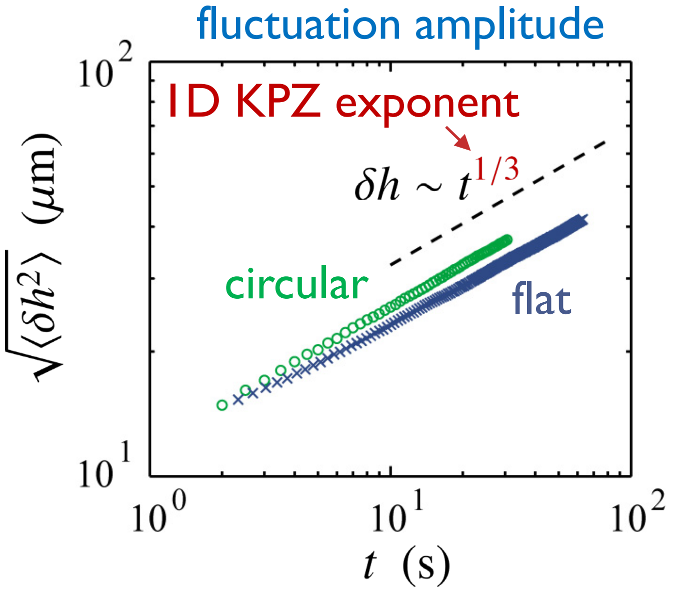


26V, 250Hz Speed x5, —200μm

We generated both circular and flat interfaces (~1000 times)

and studied interface fluctuations

Exponent & Distribution



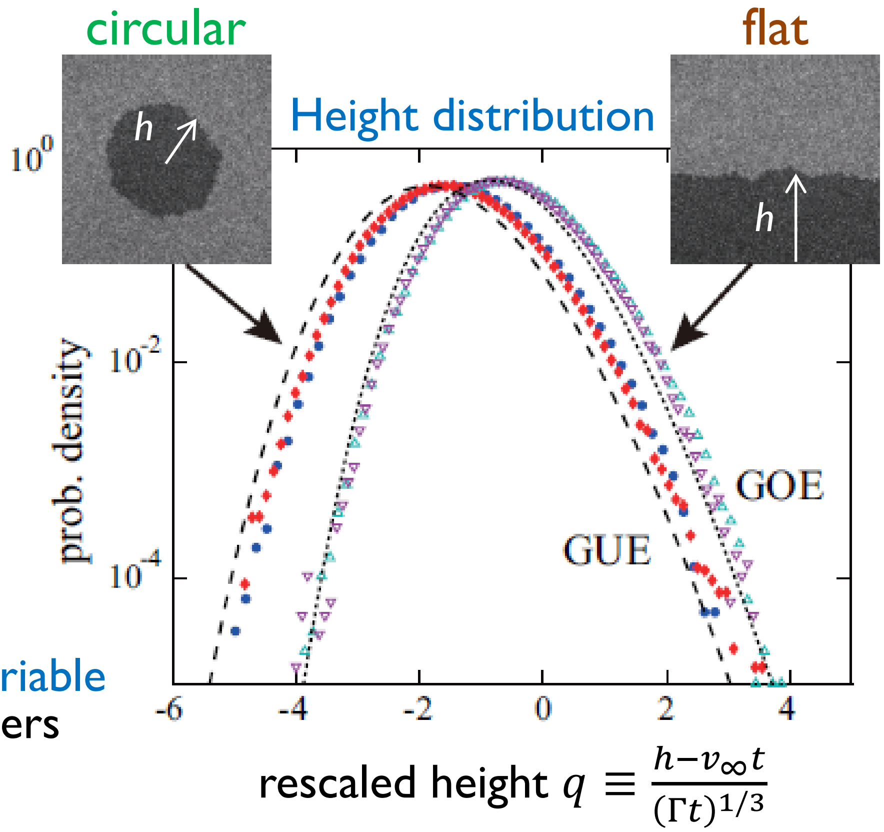
$h \simeq v_\infty t + (\Gamma t)^{1/3} \chi$

χ : rescaled variable

v_∞, Γ : parameters

circular : $h \simeq v_\infty t + (\Gamma t)^{1/3} \chi_{\mathrm{GUE}}$

flat : $h \simeq v_\infty t + (\Gamma t)^{1/3} \chi_{\mathrm{GOE}}$



- Both circular & flat cases show the same KPZ exponent (∴ KPZ class)
- Tracy-Widom distributions appeared! They are robust in experiments too!
  circular ⇒ GUE Tracy-Widom (TW) distribution flat ⇒ GOE TW dist.

More Quantitatively...

Key quantity: nth-order cumulant $\langle h^n \rangle_c$

$$\langle h^2 \rangle_c \equiv \langle \delta h^2 \rangle \qquad (\delta h \equiv h(x,t) - \langle h \rangle)$$

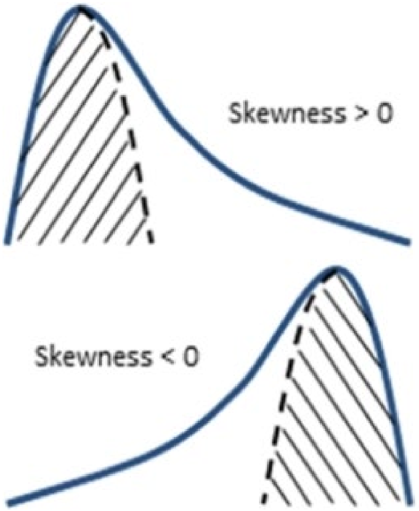
$$\langle h^3 \rangle_c \equiv \langle \delta h^3 \rangle$$

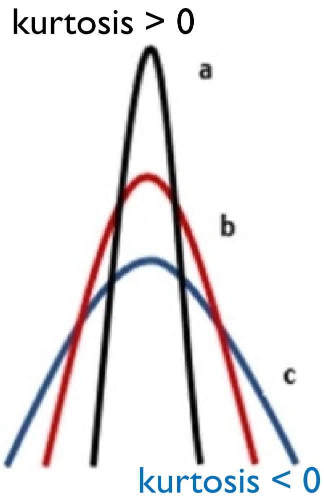
$$\langle h^4 \rangle_c \equiv \langle \delta h^4 \rangle - 3\langle \delta h^2 \rangle^2$$

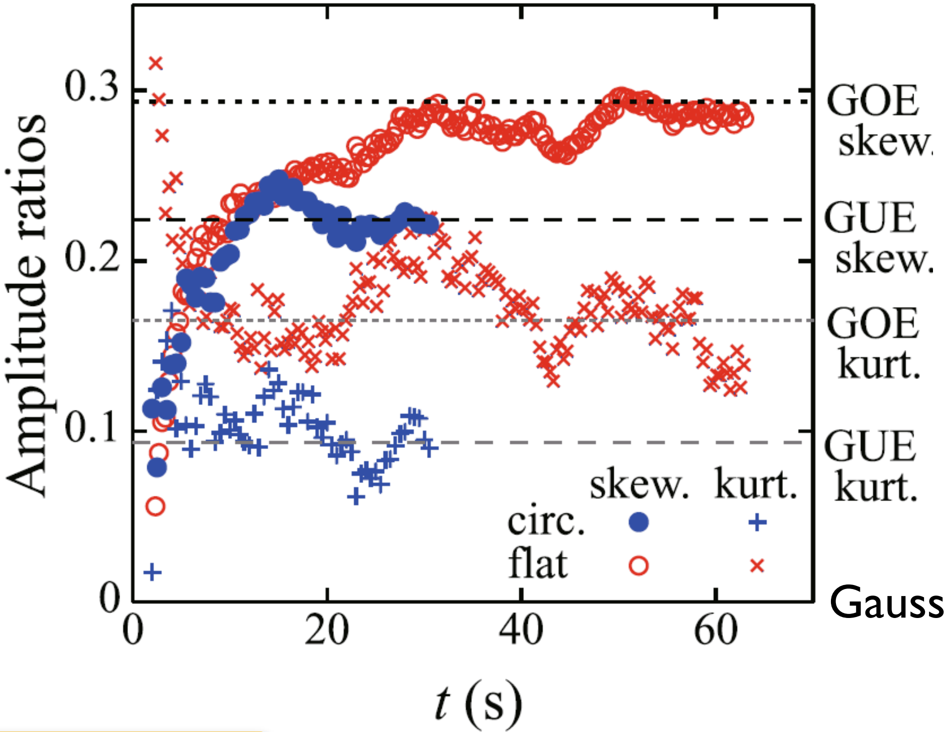
skewness $= \langle h^3 \rangle_c / \langle h^2 \rangle_c^{3/2}$

kurtosis $= \langle h^4 \rangle_c / \langle h^2 \rangle_c^{2}$

(both zero for Gaussian)





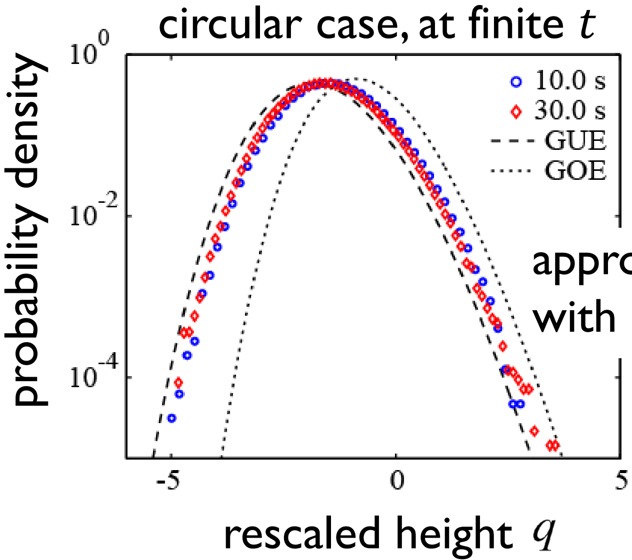


Quantitative agreement with Tracy-Widom!

Finite Time Effect

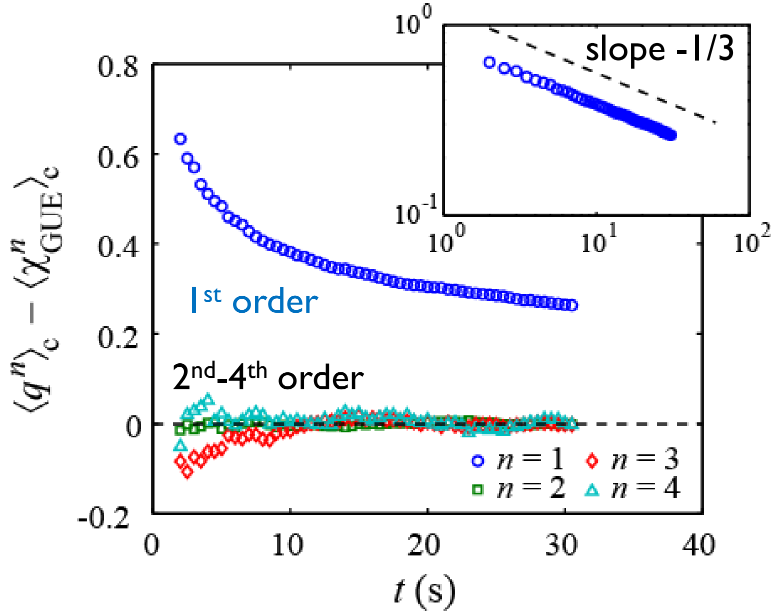
Measured rescaled height

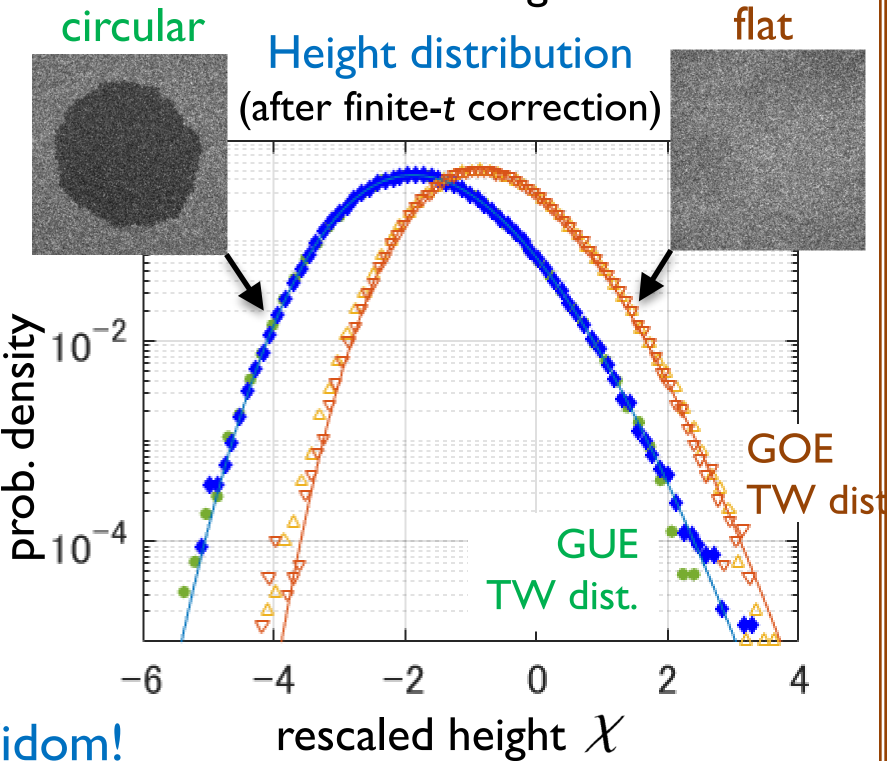
$$q \equiv (h - v_\infty t)/(\Gamma t)^{1/3} \simeq \chi$$



approaching with t

Difference from TW distribution



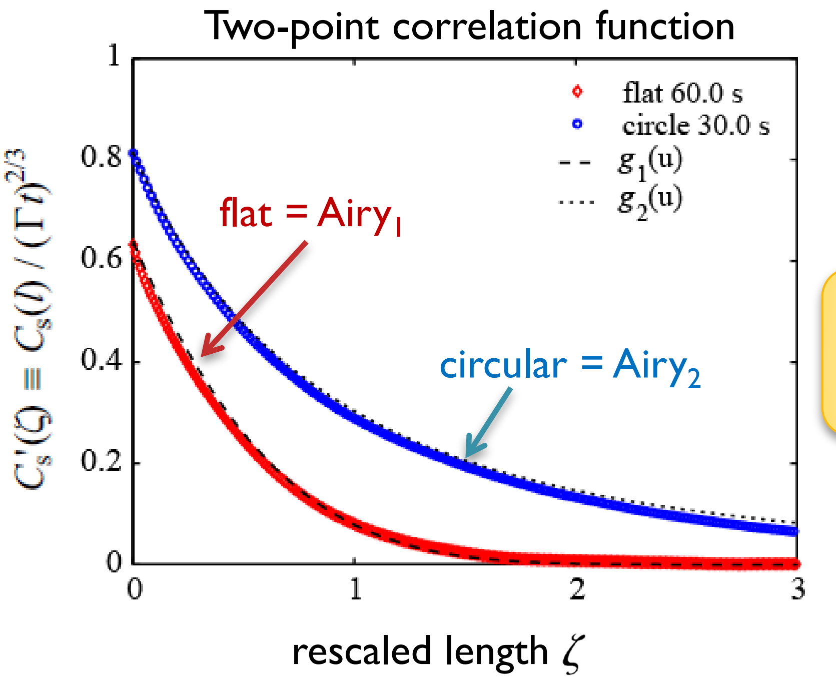


- Quantitative agreement with Tracy-Widom!
- KPZ eq approaches TW from left [29,79]. It does NOT describe this experiment!

Spatial Correlation

$\boxed{C_s(\ell,t) \equiv \langle \delta h(x+\ell,t)\delta h(x,t)\rangle}$ with $\delta h(x,t) \equiv h - \langle h \rangle$

$\overset{?}{\simeq} (\Gamma t)^{2/3} g_i\left(\frac{\ell}{\xi(t)}\right)$ with Airy 2pt correlation g_i

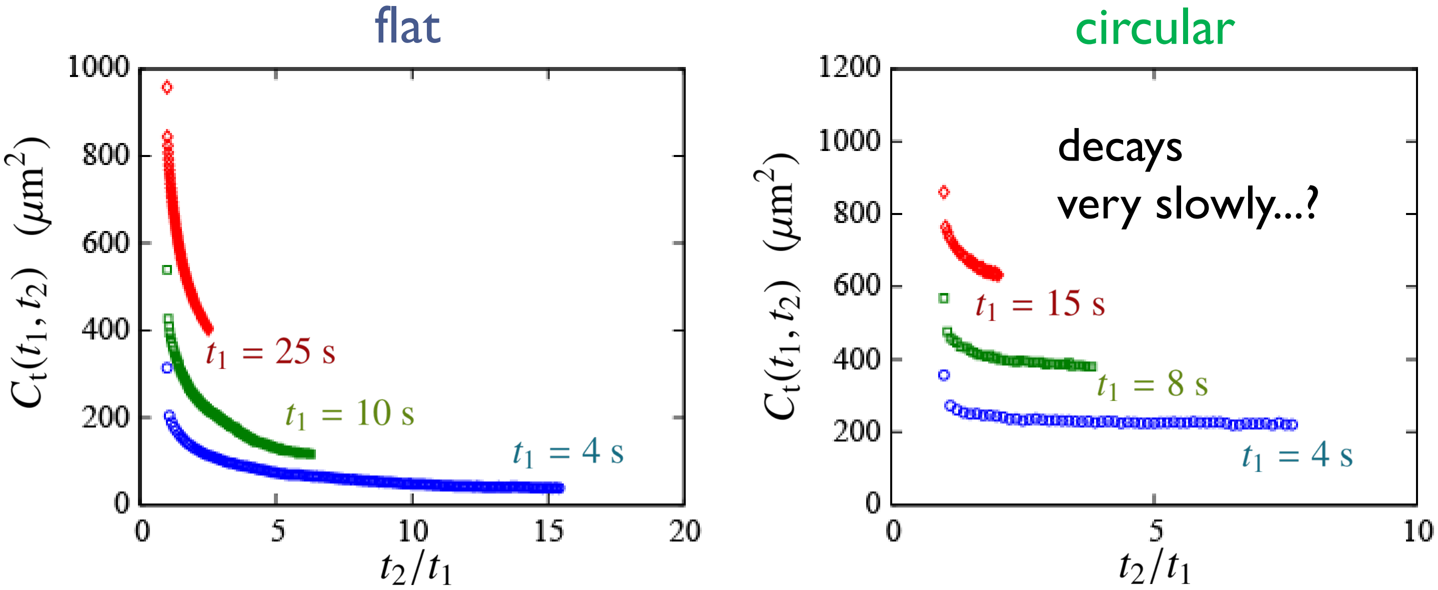

Two-point correlation function

Correlation of flat / circular interfaces is governed by the Airy_1 / Airy_2 process

Time Correlation: from Exp't to Theory

Time correlation $\boxed{C_{\mathrm{t}}(t_1, t_2) \equiv \langle \delta h(x, t_1) \delta h(x, t_2) \rangle}$ $\delta h(x, t) \equiv h(x, t) - \langle h(x, t) \rangle$

(theoretically more difficult & less understood)



decays
very slowly...?

[6,80]

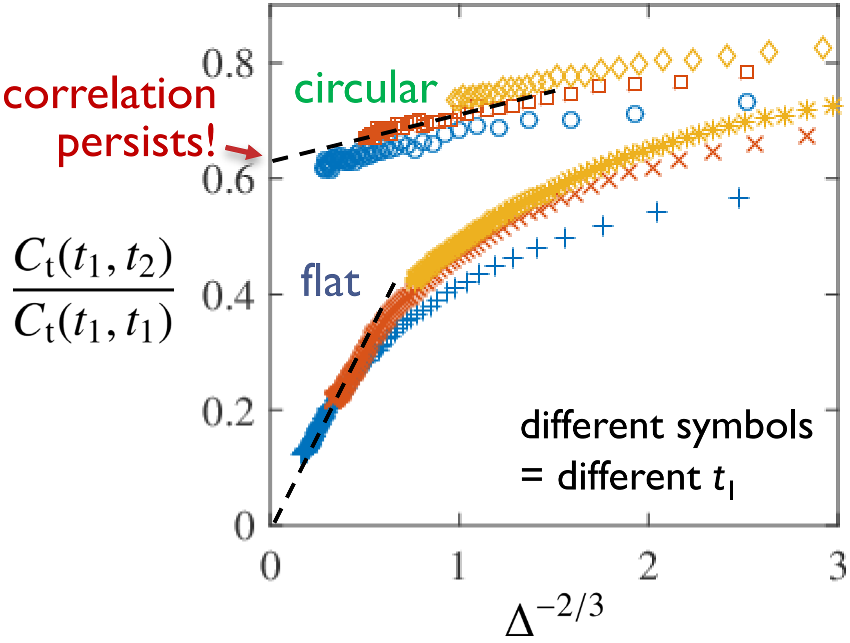
Time Correlation: from Exp't to Theory

Time corr. $\boxed{C_{\mathrm{t}}(t_1, t_2) \equiv \langle \delta h(x, t_1) \delta h(x, t_2) \rangle}$

De Nardis (left)
Le Doussal (right)
Takeuchi, 2017 [80]
(see also Ferrari & Spohn 2016 [81])

theory

experiment

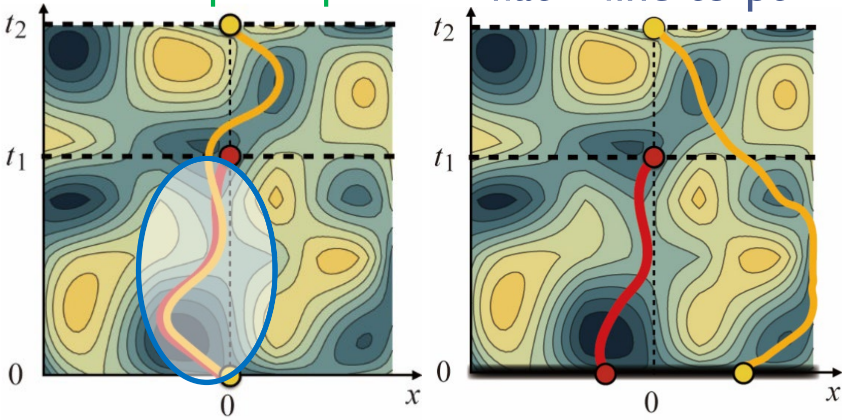


dimensionless time increment $\Delta \equiv \frac{t_2 - t_1}{t_1}$

2-time problem = 2-polymer problem

circ. = pt-to-pt

flat = line-to-pt



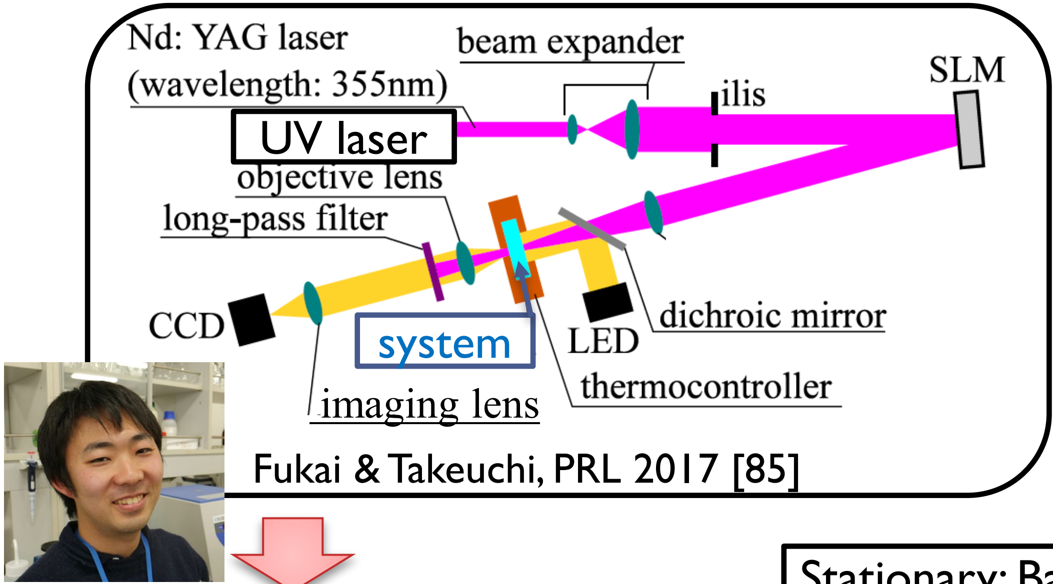
polymers overlap → correlation persists!

Approximate analytic solution was also obtained by mapping to bosons.
(see also variational approaches [81,82])

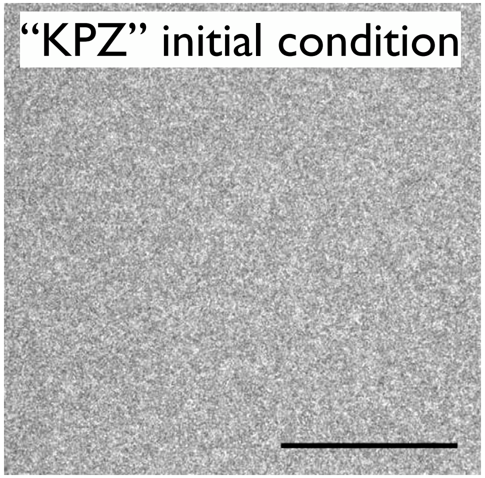
Later, 2-time correlation in circular case was solved mathematically & became a theorem.
(Johansson 2019 [83], Johansson & Rahman 2021 [84])

Exploring More Various Geometries

We can now design the initial shape arbitrarily by laser holographic technique!

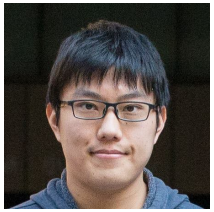


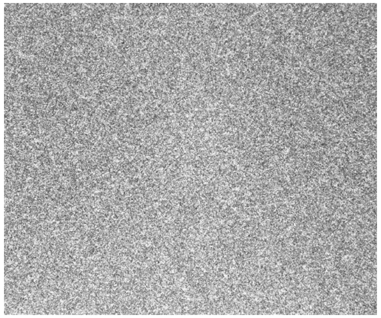
Fukai & Takeuchi, PRL 2017 [85]



realized KPZ stationary state

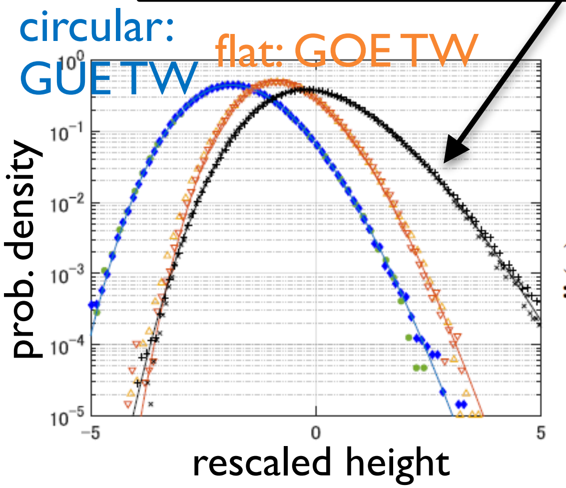
(by generating, instead of waiting)





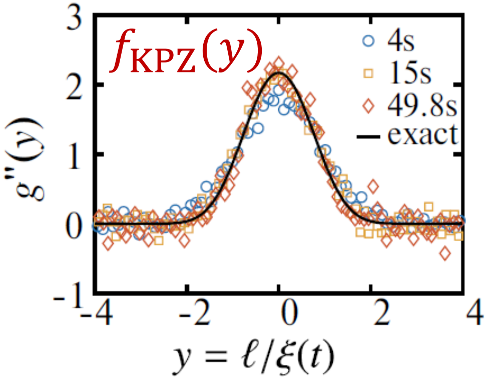
Iwatsuka et al. PRL 2020 [86]

Stationary: Baik-Rains dist.



stationary 2pt corr. func.

$\langle \frac{\partial h}{\partial x}(x+\ell, t+t_0)\frac{\partial h}{\partial x}(x,t_0)\rangle$

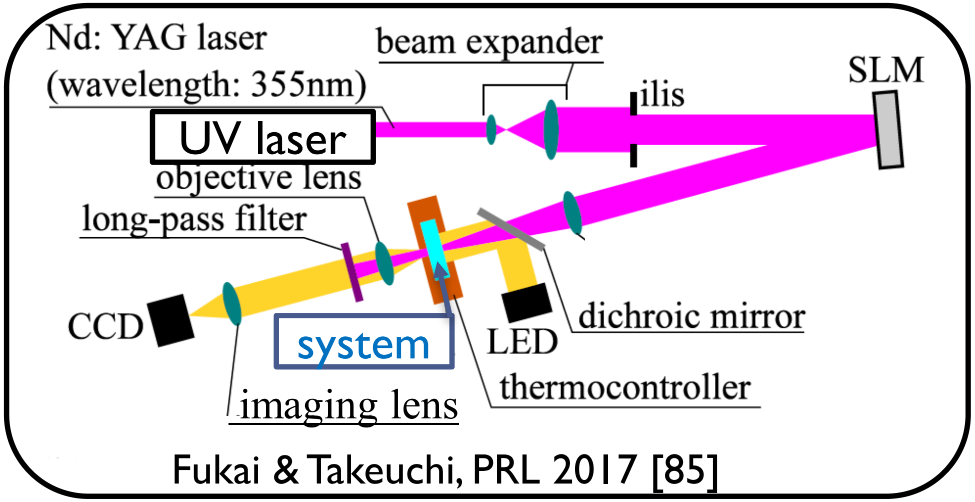


Chapter 7

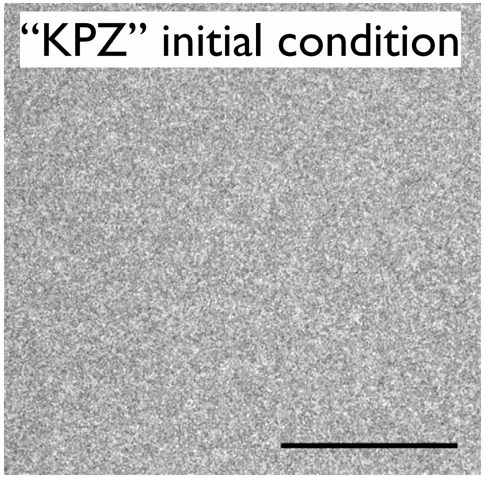
Distribution properties for general cases and variational formula

Exploring More Various Geometries

We can now design the initial shape arbitrarily by laser holographic technique!



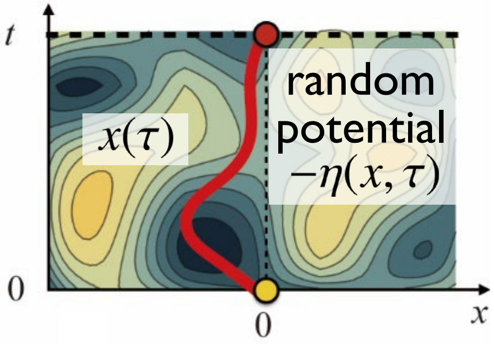
Fukai & Takeuchi, PRL 2017 [85]



- What distribution properties for such general initial conditions?
- Is there any transition/crossover between subclasses?

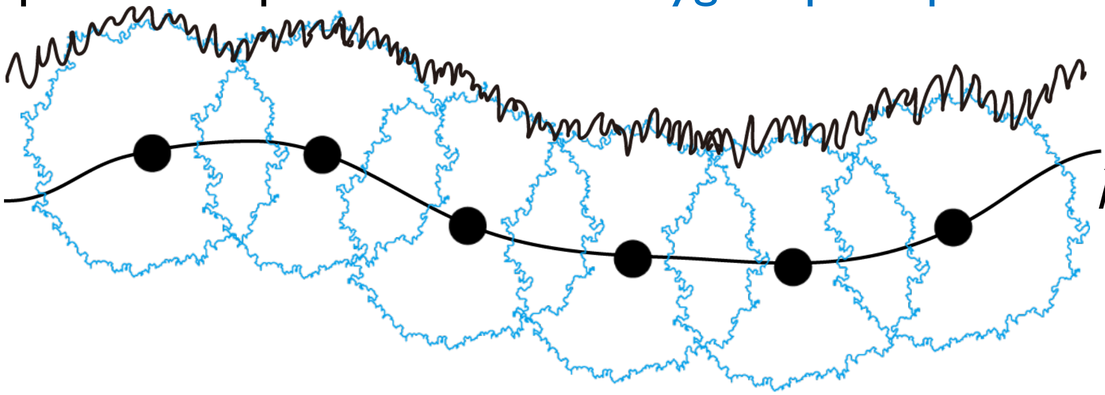
→ Powerful theoretical tool "variational formula" was proposed. [73]

Variational Formula



- Recall the directed polymer picture of KPZ eq.
  Circular case: $\underbrace{Z(x,0)=\delta(x) \to Z(x,t)}_{\text{Green function!}}$ obtained.
- $Z(x,t) \overset{\text{1pt}}{\approx} \int dy\, Z(y,0) Z_{\text{circ}}(x-y,t)$
- $$\boxed{\begin{aligned} &h(x,t) \overset{\text{1pt}}{\simeq} \sup_{y}[h_{\text{circ}}(x-y,t)+h_0(y)] \text{ with } h_0(y) \equiv h(y,0) \\ &\chi(X,t) \overset{\text{1pt}}{\simeq} \sup_{Y}\left[\mathcal{A}_2(X-Y)-(X-Y)^2+\frac{h_0(\xi(t)Y)}{(\Gamma t)^{1/3}}\right] \end{aligned}}$$

  variational formula [73]
- Graphical interpretation: "KPZ Huygens principle"



Envelope $= h(x,t)$
(only 1pt dist)

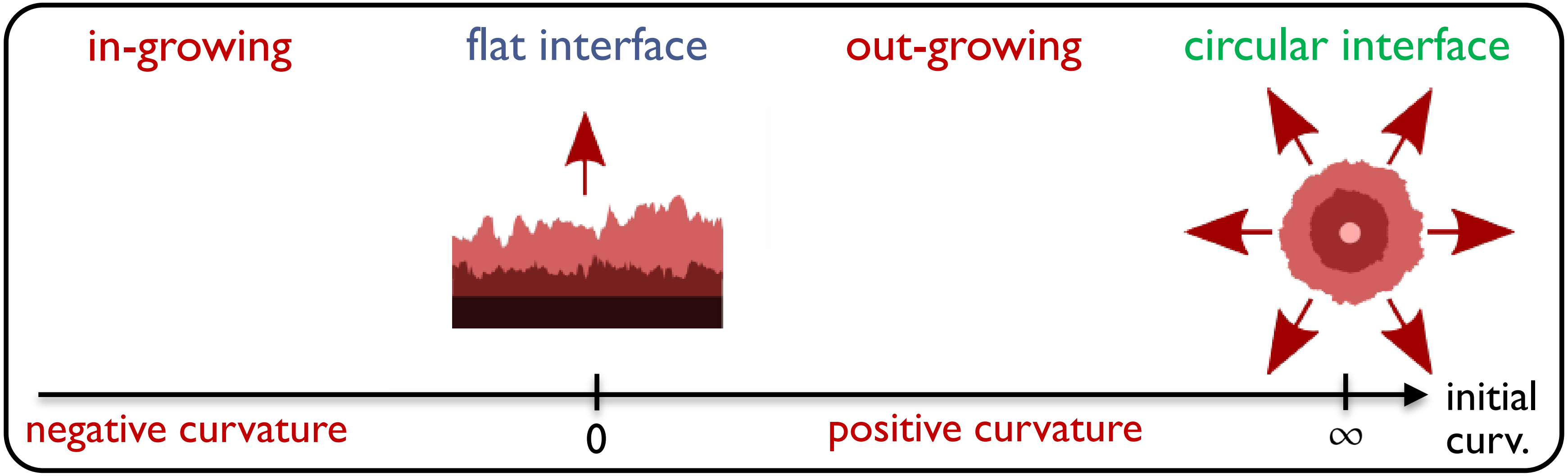
$h(x,0)$

- Method of numerical evaluation developed (Fukai & Takeuchi PRL 2020 [87])

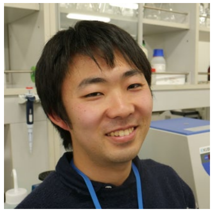
Beyond Circular & Flat Limiting Cases

- Circular interface ⇦ initially point nucleus, curvature = ∞
- Flat interface ⇦ initially straight line, curvature = 0

What happens for general initial curvature?



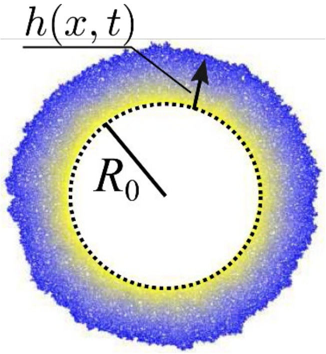
How to study? (Fukai & Takeuchi 2017 [85], 2020 [87])



- laser holography to generate an arbitrary initial condition.
- In-growing = flat statistics, then collapse [85]. How about out-growing case?

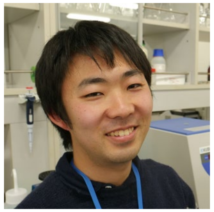
Results

(Fukai & Takeuchi PRL 2020 [87])



KPZ height $\boxed{h \simeq v_\infty t + (\Gamma t)^{1/3}\chi}$

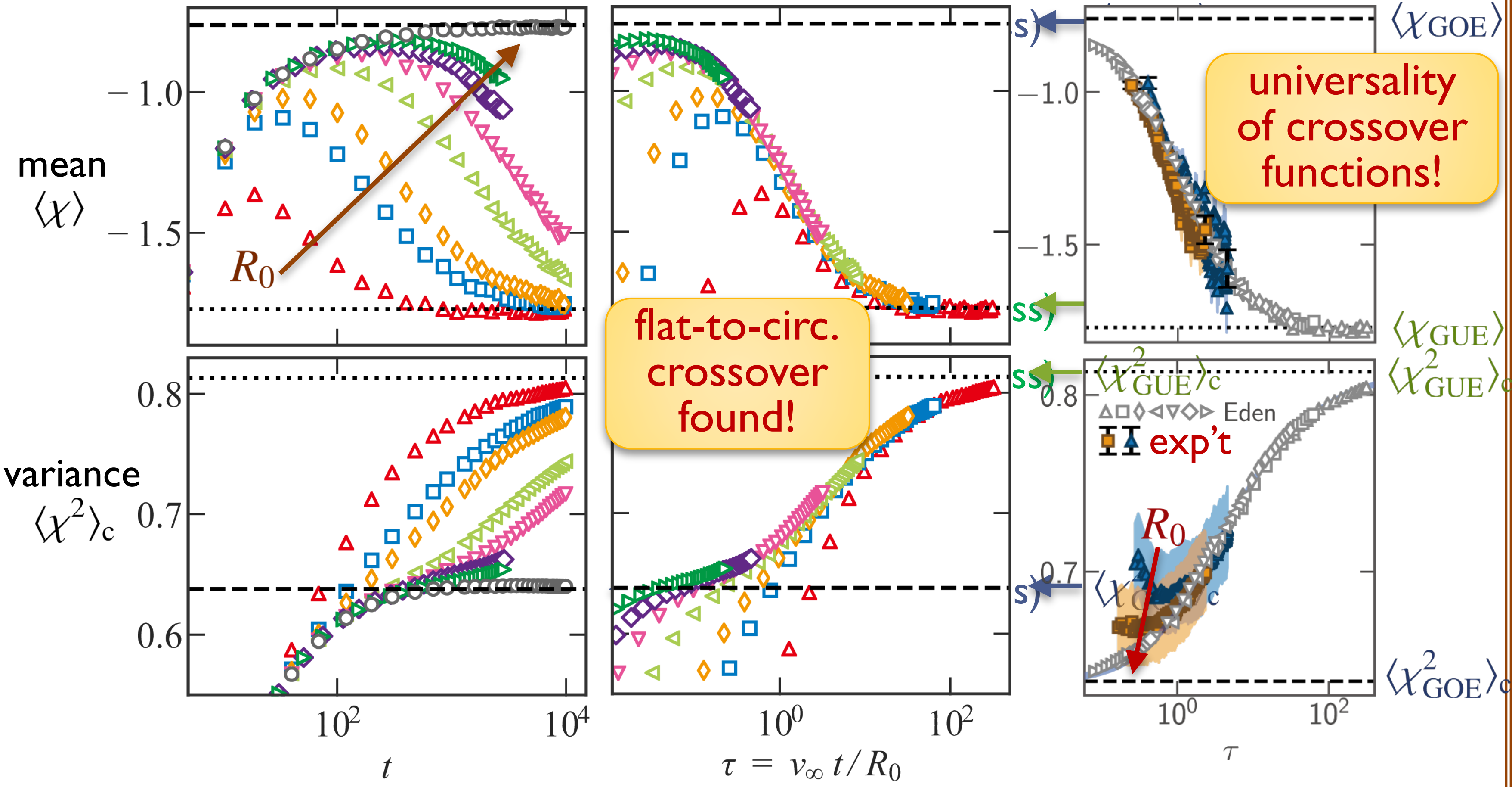
rescaled height $\left(\begin{array}{l} \chi_{\mathrm{GUE}} : \text{circ.} \\ \chi_{\mathrm{GOE}} : \text{flat} \end{array} \right)$



Eden numerics

collapse w/ rescaled time $\tau \equiv v_\infty t / R_0$

experiment

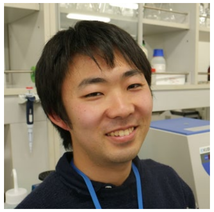


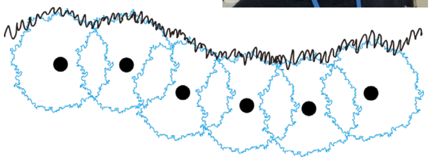
universality of crossover functions!

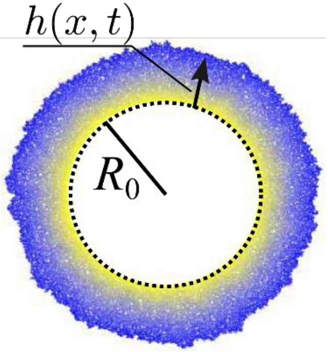
flat-to-circ. crossover found!

(Fukai & Takeuchi PRL 2020 [87])

Theoretical Account by Variational Formula







The variational formula

$$\boxed{\chi(X,t) \overset{\text{lpt}}{\simeq} \sup_Y\left[\mathcal{A}_2(X-Y)-(X-Y)^2+\frac{h_0(\xi(t)Y)}{(\Gamma t)^{1/3}}\right]}$$

here, $h_0(x) = R_0 g\left(\frac{x}{R_0}\right)$ with $g(w) \simeq 1-\frac{1}{2}w^2$

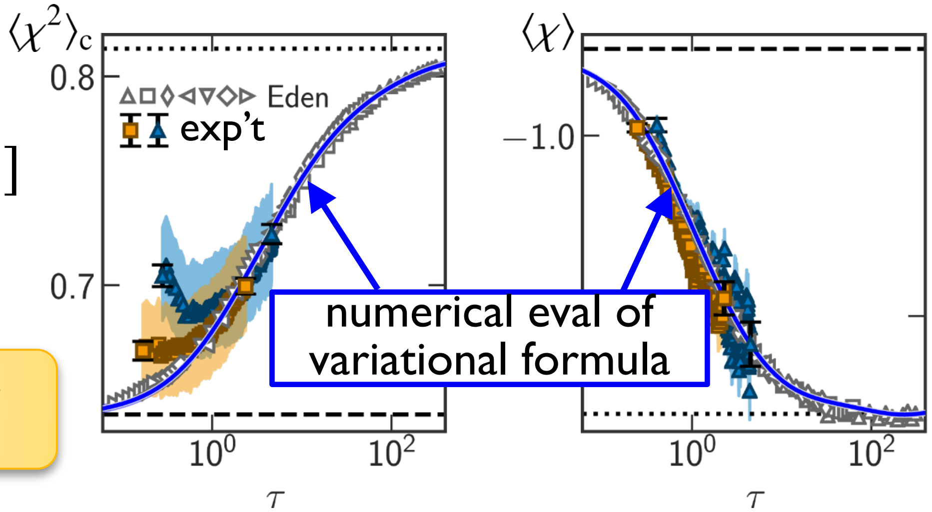
$$\therefore\ \chi(0,t) \simeq \sup_Y\left[\mathcal{A}_2(Y)-\left(1+\frac{\xi(t)^2}{2R_0(\Gamma t)^{1/3}}\right)Y^2\right]$$

$$\simeq \sup_Y[\mathcal{A}_2(Y)-(1+\tau)Y^2] \quad \left(\because\ \xi=\frac{2(\Gamma t)^{2/3}}{A}, \Gamma=\frac{1}{2}A^2v_\infty, \tau=\frac{v_\infty t}{R_0}\right)$$

$$\chi(0,t\to\infty) \simeq \mathcal{A}_2(0) = \chi_{\mathrm{GUE}}$$

$$\chi(0,t\to 0) \simeq \sup_Y[\mathcal{A}_2(Y)-Y^2]$$

$$= \chi_{\mathrm{GOE}}$$ [71,88]



numerical eval of variational formula

Variational formula fully accounts for the flat-circular crossover!

As a Final Remark...

Equilibrium (major player: Ising)

1869	Discovery of liquid-vapor critical point (which is Ising)
1890's-	$\beta \approx 0.3\text{-}0.4$ (cf. 3D Ising $\beta \approx 0.326$)
1944	Onsager's solution to 2D Ising
1950's-	Experiments on binary fluids & Ising-type magnets
1971	Wilson's renormalization group, ϕ^4 model (continuum equation) "Ising universality class"
1984	2D conformal field theory classifying universality classes
2011-	Conformal approach to 3D Ising
⋮	⋮

Non-eq (major player: KPZ?)

1980's	Scaling laws for discrete models of interface growth
1986	KPZ eq. (continuum eq.)
1997	Experiments on KPZ exponents
2000	Exact solutions to 1D discrete models
2010	Experiment on exact results
2010	Exact solutions to 1D KPZ eq.
2017	Discussion started on relation to isotropic Heisenberg spin chain
2019	KPZ corr. func. in Heisenberg
2021-22	Exp'ts on KPZ-Heisenberg link
⋮	⋮

Can we expect as glorious breakthroughs for KPZ? Even more surprises?

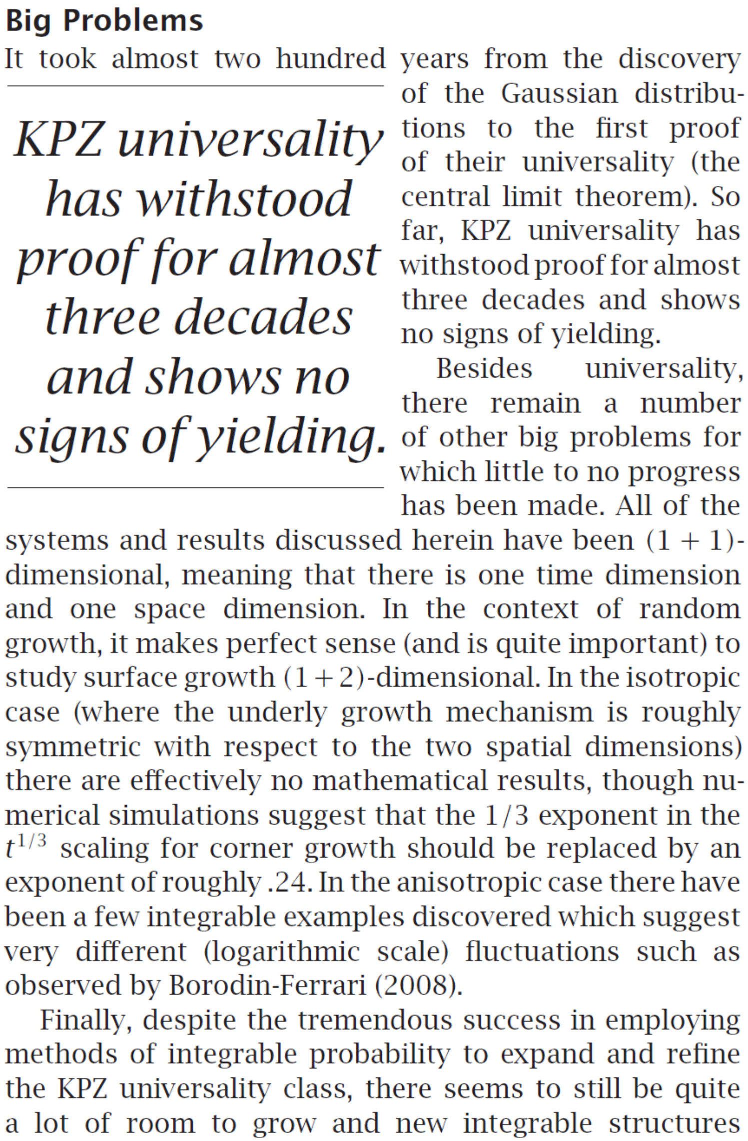
As a Final Remark...

Another perspective:

A strongly correlated version of the Central Limit Theorem?

higher dimension?

Corwin's review 2016 [89]



Big Problems

It took almost two hundred years from the discovery of the Gaussian distributions to the first proof of their universality (the central limit theorem). So far, KPZ universality has withstood proof for almost three decades and shows no signs of yielding.

KPZ universality has withstood proof for almost three decades and shows no signs of yielding.

Besides universality, there remain a number of other big problems for which little to no progress has been made. All of the systems and results discussed herein have been (1 + 1)-dimensional, meaning that there is one time dimension and one space dimension. In the context of random growth, it makes perfect sense (and is quite important) to study surface growth (1 + 2)-dimensional. In the isotropic case (where the underly growth mechanism is roughly symmetric with respect to the two spatial dimensions) there are effectively no mathematical results, though numerical simulations suggest that the 1/3 exponent in the $t^{1/3}$ scaling for corner growth should be replaced by an exponent of roughly .24. In the anisotropic case there have been a few integrable examples discovered which suggest very different (logarithmic scale) fluctuations such as observed by Borodin-Ferrari (2008).

Finally, despite the tremendous success in employing methods of integrable probability to expand and refine the KPZ universality class, there seems to still be quite a lot of room to grow and new integrable structures

Can we expect as glorious breakthroughs for KPZ? Even more surprises?